Compact Grid Representation of Graphs*

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Abstract. A graph G is said to be *grid locatable* if it admits a representation such that vertices are mapped to grid points and edges to line segments that avoid grid points but the extremes. Additionally G is said to be *properly embeddable in the grid* if it is grid locatable and the segments representing edges do not cross each other. We study the area needed to obtain those representations for some graph families.

Keywords: graph drawing, grid locatable, grid embeddable, chromatic number.

1 Introduction

Graph drawing applies topology and geometry to derive suitable representations of graphs. Particularly, grid representations of graphs have attracted the attention of many researchers (see, for example [4,7,11]).

A *grid point* is a point of the plane having integer coordinates. A line segment s joining two grid points is said to be *primitive* if the only grid points in s are its extremes.

A graph G is said to be *grid locatable* (or *locatable in the grid*), if each vertex is represented by a grid point and each edge by a primitive segment joining its extremes. G is said to be *properly embeddable in the grid* (*p–embeddable* for short) if it is grid locatable and the segments representing edges do not cross each other. Observe, that the graphs so obtained are subgraphs of the visibility graph of the grid points, where two points are visible if there is no other grid point between them (this visibility graph was defined, for example, in [12]).

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In this way, a graph is grid locatable (resp. p-embeddable) if it is isomorphic to a subgraph (resp. plane subgraph) of the visibility graph of the grid points.

The following characterization is shown in [9].

Theorem 1. *[9] A graph G is grid locatable if and only if G is 4-colorable.*

One of the reasons to study grid representations of graphs is to test how compact a given representation can be. Thus, a very important question in this kind of representation is to minimize the area needed to represent a given graph.

A *line* of the grid $\mathbb{Z} \times \mathbb{Z}$ is a set $\{(x, k)/x \in \mathbb{Z}\}$ for $k \in \mathbb{Z}$ fixed. By grid locating (or p-embedding) a graph G *in ℓ lines* we mean to locate (or embed) G in $\mathbb{Z} \times \mathbb{Z}$ mapping the vertices to grid points in ℓ *consecutive* lines (there is no unused line between two lines in our drawing).

In [9], an upper bound on the number of lines needed to grid locate a graph in the grid is given.

Corollary 1. *[9] A graph G with chromatic number $\chi(G) \leq 4$ can be located in the grid in, at most, $\chi(G)$ lines.*

Although the authors give an upper bound to the number of lines needed in order to grid locate a graph with $\chi(G) \leq 4$, in general, that upper bound is not tight. In this work, we deal with this problem. Namely, to find a tighter upper bound of the area (number of lines) needed to grid locate or p-embed a given graph in the grid.

A simple but useful observation is the following:

Remark 1. An edge joining two grid points (x_1, y_1) and (x_2, y_2) is primitive if and only if $|x_1 - x_2|$ and $|y_1 - y_2|$ are relatively prime to each other.

The structure of this paper is as follows. In Sect. 2, we show some results about p-embeddability of graphs in the grid. In Sect. 3, we provide our main results, giving a characterization of those graphs that are grid locatable in two and three lines. We shall conclude with some remarks and open problems.

2 P-Embedding a Graph in the Grid

Until very recently, it was not known whether any planar graph could be p-embedded in the plane or not, and there were only some partial results on the subject (see [9]). This problem has just been solved in the affirmative independently by M. Balko [1] and Flores-Peñaloza et al. [8], although the grid size of the resulting representation is still excessively large. So it makes sense to find bounds on the size of the grid for some specific families of graphs, as we do below for bipartite graphs.

Proposition 1. *Any plane bipartite graph can be p-embedded in the grid.*