Chapter 5
Configurational Perturbations of Energy Functionals

The evaluation of the topological derivatives for the energy shape functionals associated to the representative boundary value problems for the scalar (Laplace) and the vectorial (Navier) second-order partial differential equations and for the scalar fourth-order (Kirchhoff) partial differential equation is presented in this chapter. In contrast with Chapter 4 here the domain is topologically perturbed by the nucleation of a small inclusion, instead of a hole.

More precisely, the perturbed domain is obtained if a circular hole $B_\varepsilon(\hat{x})$ is introduced inside $\Omega \subset \mathbb{R}^2$, where $B_\varepsilon(\hat{x}) \subseteq \Omega$ denotes a ball of radius $\varepsilon$ and center at $\hat{x} \in \Omega$. Then, $B_\varepsilon(\hat{x})$ is filled by an inclusion with different material property compared to the unperturbed domain $\Omega$, as it shown in fig. 5.1. The material properties are characterized by a piecewise constant function $\gamma_\varepsilon$ of the form

$$
\gamma_\varepsilon = \gamma_\varepsilon(x) := \begin{cases} 
1 & \text{if } x \in \Omega \setminus B_\varepsilon, \\
\gamma & \text{if } x \in B_\varepsilon,
\end{cases}
$$

(5.1)

where $\gamma \in \mathbb{R}_+$ is the contrast coefficient.

In the same way as in Chapter 4 the shape change velocity field $\mathcal{V} \in \mathcal{S}_\varepsilon$ that represents an uniform expansion of the circular inclusion $B_\varepsilon(\hat{x})$ is constructed, with the set $\mathcal{S}_\varepsilon$ defined in (4.2). Such a velocity field $\mathcal{V}$ is the key point when using Proposition 1.1 leading to a simple and constructive method for evaluation of the topological derivative through formula (1.49). Note that in this case the topologies of the original and perturbed domains are preserved. However, we are introducing a nonsmooth perturbation in the coefficients of the differential operator through the contrast $\gamma_\varepsilon$, by changing the material property of the background in a small region $B_\varepsilon \subset \Omega$. This procedure is called the configurational perturbation. Since we are dealing with a nonsmooth perturbation of the material properties, the sensitivity of the shape functional with respect to the nucleation of an inclusion can also be studied through the topological asymptotic analysis concept, which is, in fact, the most appropriate approach for such a problem.
5.1 Second Order Elliptic Equation: The Laplace Problem

In this section we evaluate the topological derivative of the total potential energy associated to the steady-state heat conduction problem, considering the nucleation of a small inclusion, represented by $B_\varepsilon \subset \Omega$, as the topological perturbation.

5.1.1 Problem Formulation

The shape functional associated to the unperturbed domain which we are dealing with is defined as

$$\psi(\chi) := \mathcal{J}_\Omega(u) = -\frac{1}{2} \int_{\Omega} q(u) \cdot \nabla u + \int_{\Gamma_N} q^\ast u, \quad (5.2)$$

where the scalar function $u$ is the solution to the variational problem:

$$\left\{ \begin{array}{l}
\text{Find } u \in \mathcal{U}, \text{ such that } \\
\int_{\Omega} q(u) \cdot \nabla \eta = \int_{\Gamma_N} q^\ast \eta \quad \forall \eta \in \mathcal{V}, \\
\text{with } q(u) = -k \nabla u.
\end{array} \right. \quad (5.3)$$

In the above equation, $k$ is the thermal conductivity of the medium, assumed to be constant everywhere. The set $\mathcal{U}$ and the space $\mathcal{V}$ are respectively defined as

$$\mathcal{U} := \{ \varphi \in H^1(\Omega) : \varphi|_{\Gamma_D} = \overline{u} \}, \quad (5.4)$$

$$\mathcal{V} := \{ \varphi \in H^1(\Omega) : \varphi|_{\Gamma_D} = 0 \}. \quad (5.5)$$

In addition, $\partial \Omega = \Gamma_D \cup \Gamma_N$ with $\Gamma_D \cap \Gamma_N = \emptyset$, where $\Gamma_D$ and $\Gamma_N$ are Dirichlet and Neumann boundaries, respectively. Thus $\overline{u}$ is a Dirichlet data on $\Gamma_D$ and $\overline{q}$ is a