Chapter 6
Topological Derivative Evaluation with Adjoint States

The evaluation of the topological derivative for a general class of shape functionals is presented in this chapter. The method is applied to a modified energy shape functional associated with the steady-state heat conduction problem. The nucleation of a small circular hole, represented by $B_\varepsilon(\hat{x})$, with $\hat{x} \in \Omega \subset \mathbb{R}^2$ and $\overline{B_\varepsilon} \in \Omega$, is considered as the topological perturbation. Therefore, the topologically perturbed domain is obtained as $\Omega_\varepsilon = \Omega \setminus \overline{B_\varepsilon}$.

For the specific problem, some methods were proposed to evaluate the topological derivative $[11, 130, 184]$. In this chapter, we extend the approach of Chapter 4 (see also [184]) to deal with the modified adjoint method proposed in [11]. This leads to an alternative approach to calculate the topological derivative based on shape sensitivity analysis combined with the modified Lagrangian method.

Since we consider a general class of shape functionals, which are not necessarily associated with the energy, we will show later that the proposed approach simplifies the most delicate step of the topological derivative calculation, namely, the asymptotic analysis of the adjoint state.

6.1 Problem Formulation

The shape functional in the unperturbed domain which we are dealing with is defined as

$$\psi(\chi) := \mathcal{J}_\Omega(u) = \frac{1}{2} \int_\Omega B\nabla u \cdot \nabla u,$$

where $B$ is a given second order symmetric constant tensor and the scalar function $u$ is the solution to the variational problem:

$$\begin{cases}
\text{Find } u \in \mathcal{U}, \text{ such that } \\
\int_{\Omega} \nabla u \cdot \nabla \eta + \int_{\Gamma_N} \overline{q} \eta = 0 \quad \forall \eta \in \mathcal{V}.
\end{cases}$$
The set $\mathcal{U}$ and the space $\mathcal{V}$ are respectively defined as
\begin{equation}
\mathcal{U} := \{ \phi \in H^1(\Omega) : \phi|_{\Gamma_D} = \overline{\pi} \}, \tag{6.3}
\end{equation}
\begin{equation}
\mathcal{V} := \{ \phi \in H^1(\Omega) : \phi|_{\Gamma_D} = 0 \}. \tag{6.4}
\end{equation}

In addition, $\partial \Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$ with $\overline{\Gamma}_D \cap \overline{\Gamma}_N = \emptyset$, where $\overline{\Gamma}_D$ and $\overline{\Gamma}_N$ are Dirichlet and Neumann boundaries, respectively. Thus $\overline{\pi}$ is a Dirichlet data on $\overline{\Gamma}_D$ and $\overline{\eta}$ is a Neumann data on $\overline{\Gamma}_N$, both assumed to be smooth enough. See the details in fig. 6.1. The strong equation associated to the above variational problem (6.2) reads:
\begin{equation}
\begin{cases}
\text{Find } u, \text{ such that } \\
-\Delta u = 0 \text{ in } \Omega, \\
u = \overline{\pi} \text{ on } \Gamma_D, \\
-\partial_n u = \overline{\eta} \text{ on } \Gamma_N.
\end{cases} \tag{6.5}
\end{equation}

Remark 6.1. The functional (6.1) includes a large range of shape functions, which shall be useful for practical applications. In particular, when $B = I$, the functional (6.1) degenerates to the energy. In addition, when $B \neq I$, the analysis becomes much more involved, which justifies the introduction of a modified adjoint state, as already mentioned in the beginning of this chapter.

Now, let us state the same problem in the perturbed domain. In this case, the shape functional reads
\begin{equation}
\psi(\chi_\varepsilon) := \mathcal{J}_{\varepsilon}(u_\varepsilon) = \frac{1}{2} \int_{\Omega_\varepsilon} B \nabla u_\varepsilon \cdot \nabla u_\varepsilon, \tag{6.6}
\end{equation}
where the scalar function $u_\varepsilon$ solves the variational problem:
\begin{equation}
\begin{cases}
\text{Find } u_\varepsilon \in \mathcal{U}_\varepsilon, \text{ such that } \\
\int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla \eta + \int_{\Gamma_N} \overline{\eta} \eta = 0 \quad \forall \eta \in \mathcal{V}_\varepsilon.
\end{cases} \tag{6.7}
\end{equation}