Forcing Axioms, Finite Conditions and Some More

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Abstract. We survey some classical and some recent results in the theory of forcing axioms, aiming to present recent breakthroughs and interest the reader in further developing the theory. The article is written for an audience of logicians and mathematicians not necessarily familiar with set theory.

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1 Introduction

We shall work within the axioms of the Zermelo-Fraenkel set theory with Choice (ZFC). These axioms were introduced basically starting from 1908 and improving to a final version in the 1920s as an attempt to axiomatize the foundations of mathematics. There have been other such attempts at about the same time and later, but it is fair to say that for the purposes of much of modern mathematics the axioms of ZFC represent the accepted foundation (see [13] for a detailed discussion of foundational issues in set theory). Gödel’s Incompleteness theorems [16] prove that for any consistent theory $T$ which implies the Peano Axioms and whose axioms are presentable as a recursively enumerable set of sentences, so for any reasonable theory one would say, there is a sentence $\varphi$ in the language of $T$ such that $T$ does not prove or disprove $\varphi$. In some sense the discussion of which axioms to use is made less interesting by these theorems, which can be interpreted as saying that a perfect choice of axioms does not exists. We therefore do like the most, we concentrate on the axioms that correctly model most of mathematics, and for the rest, we try to understand the limits and how we can improve them. For us ZFC is a basis for a foundation which in some circumstances can be extended to a larger set of axioms which provide an insight into various parts of mathematics. In here we concentrate on the forcing axioms (and their negations).

2 The Discovery of Forcing

The proof of Gödels’ Incompleteness theorems is not constructive and in particular it does not construct an independent sentence $\varphi$, it only proves its existence.

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It is therefore quite amazing that for the theory of ZFC such a sentence \( \varphi \) turned out to be the following simple statement formulated by Cantor as early as 1878 [8] (as an implicit conjecture only at that point):

**Continuum Hypothesis (CH):** For every infinite subset \( A \) of the reals \( \mathbb{R} \), either there is a bijection between \( A \) and \( \mathbb{R} \) or there is a bijection between \( A \) and \( \mathbb{N} \).

This statement tormented Cantor, who could not prove it or disprove it. With a good reason, since it was finally proved by Cohen in [9] that if ZFC is consistent then so is ZFC with the negation of CH. Since Gödel [17] had proved that if ZFC is consistent then so is ZFC along with CH, it follows that CH is independent of ZFC. To obtain his proof Cohen introduced the technique of forcing. It is a technique to extend a universe \( V \) of set theory to another one, \( V[G] \), so that

- has the same ordinals
- (most often) has the same cardinals and
- satisfies a desired formula \( \varphi \).

One way to think of this technique is to imagine that we are actually working within some large ambient model of ZFC and seeing only a small submodel which we call \( V \). This submodel may even be assumed to be countable. Being so small, \( V \) has a rather particular opinion of the reality, for example it esteem that every infinite cardinal \( \aleph_\alpha \) is some ordinal \( \beta(\aleph_\alpha) \) among the ordinals \( \beta \) that actually belong to \( V \) (we denote this by \( \aleph_\alpha^V \)). For Cohen’s proof we may also assume that \( V \) satisfies CH- since if it does not we have already violated CH. What we aim to do is to extend \( V \) to a larger model which will contain \( \aleph_2^V \) many reals from our ambient universe, while \( V[G] \) and \( V \) will actually agree on their opinion of what is \( \aleph_1 \) and \( \aleph_2 \) (they will have the same cardinals). Then in \( V[G] \) we can choose any set \( A \) of only \( \aleph_1^V \) reals to demonstrate that \( A \) is not bijective with either \( \mathbb{N} \) or \( \mathbb{R} \), hence CH fails. This construction rests upon a combinatorial method which adds these new reals while preserving the cardinals. We may imagine this as a sort of inductive construction, but one in which the desired object is not added using a linearly ordered set of approximations but rather a partially ordered set.

For example, thinking of a real as a function from \( \omega \) to 2 (as there is a bijection between \( \mathbb{R} \) and \( \mathcal{P}(\omega) \)), we may add a real by considering the partial order of finite partial functions from \( \omega \) to 2 in their increasing order, some coherent subset of which will be glued together to give us a total function from \( \omega \) to 2. The coherent subset is our \( G \), the generic filter. The fact that such a subset can be chosen is one of the main ingredients of the method. The actual proof of the negation of CH requires us to work with functions from \( \omega_2 \times \omega \) to 2, but the idea is the same.

Partial orders considered in the theory of forcing have the property of having the smallest element and are often called **forcing notions**. Elements of a forcing notion are usually called **conditions**. As we are looking for coherent subsets of a forcing notion, an important point is to consider for given two conditions if they are coherent, which means that they have a common extension. We say that conditions having such an extension are **compatible**, otherwise they are **incompatible**. A set of conditions is called an **antichain** if it consists of pairwise