Linear Vertex-kernels for Several Dense Ranking $r$-Constraint Satisfaction Problems

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Abstract. A ranking $r$-constraint satisfaction problem (ranking $r$-CSP for short) consists of a ground set of vertices $V$, an arity $r \geq 2$, a parameter $k \in \mathbb{N}$ and a constraint system $c$, where $c$ is a function which maps rankings (i.e. orderings) of $r$-sized sets $S \subseteq V$ to $\{0, 1\}$ [16]. The objective is to decide if there exists a ranking $\sigma$ of the vertices satisfying all but at most $k$ constraints (i.e. $\sum_{S \subseteq V, |S| = r} c(\sigma(S)) \leq k$).

Famous ranking $r$-CSPs include Feedback Arc Set in Tournaments and Dense Betweenness [4,15]. In this paper, we prove that so-called $l_r$-simply characterized ranking $r$-CSPs admit linear vertex-kernels whenever they admit constant-factor approximation algorithms. This implies that $r$-Dense Betweenness and $r$-Dense Transitive Feedback Arc Set [15], two natural generalizations of the previously mentioned problems, admit linear vertex-kernels. Both cases were left opened by Karpinski and Schudy [16]. We also consider another generalization of Feedback Arc Set in Tournaments for constraints of arity $r \geq 3$, that does not fit the aforementioned framework. Based on techniques from [11], we obtain a 5-approximation and then provide a linear vertex-kernel. As a main consequence of our result, we obtain the first constant-factor approximation algorithm for a particular case of the so-called Dense Rooted Triplet Inconsistency problem [9].

1 Introduction

Parameterized complexity is a powerful theoretical framework to cope with NP-Hard problems. The aim is to identify some parameter $k$ independent from the instance size $n$, which captures the exponential growth of the complexity to solve the problem at hand. A parameterized problem is said to be fixed parameter tractable whenever it can be solved in $f(k) \cdot n^{O(1)}$ time, where $f$ is any computable function [12][18]. In this paper, we focus on kernelization. A kernelization algorithm (or kernel for short) for a parameterized problem $\Pi$ is a polynomial-time algorithm that given an instance $(I, k)$ of $\Pi$ outputs an equivalent instance $\langle I', k' \rangle$ of $\Pi$ such that $|I'| \leq g(k)$ and $k' \leq k$. The function $g$ is said to be the size of the kernel, and $\Pi$ admits a polynomial kernel whenever $g$ is a polynomial. A well-known result states that a (decidable) parameterized problem is fixed parameter tractable if and only if it admits a kernel [18]. Observe
that this result provides kernels of super-polynomial size. Recently, several results gave evidence that some parameterized problems do not admit polynomial kernels (under complexity-theoretic assumptions [7,8]).

We mainly study ranking \( r \)-CSPs from the kernelization viewpoint. In a ranking \( r \)-CSP, a ground set of vertices \( V \), an arity \( r \geq 2 \) and a set of constraints defined on \( r \)-sized subsets \( S \subseteq V \) are given. Here, a constraint corresponds to some allowed rankings on \( S \). The aim of such problems is to find a linear ranking on \( V \) that minimizes the number of constraints ranked in a non allowed manner. We study the decision version of such problems, where the instance comes together with some parameter \( k \in \mathbb{N} \) and the aim is to decide if there exists a ranking satisfying all but at most \( k \) constraints. We consider such problems on dense instances, where every set of \( r \) vertices is a constraint. For instance, Feedback Arc Set in Tournaments fits this framework with \( r = 2 \), any arc \( uv \) being satisfied by a ranking \( \sigma \) iff \( u <_\sigma v \). Such problems can be equivalently stated in terms of editing problems: can we edit at most \( k \) constraints to obtain an instance that admits a ranking satisfying all its constraints?

**Related Results.** While a lot of kernelization results are known for graph editing problems [6,17,20,21], fewer results exist regarding directed graph and hypergraph editing problems. An example of polynomial kernel for a directed graph editing problem is the quadratic vertex-kernel for Transitivity Editing [22]. Regarding dense ranking \( r \)-CSPs, Feedback Arc Set in Tournaments and Dense Betweenness are NP-Complete [23,10] but fixed parameter tractable [4,15], and both admit a linear vertex-kernel [5,19]. Recently, Karpinski and Schudy [16] showed PTASs and subexponential parameterized algorithms for (weakly)-fragile ranking \( r \)-CSPs. A constraint is (weakly-)fragile if whenever it is satisfied by one ranking then making one single move (resp. making one of the following moves: swapping the first two vertices, the last two vertices or making a cyclic move) makes it unsatisfied.

**Our Results.** We introduce so-called \( l_r \)-simply characterized ranking \( r \)-CSPs, and prove that such problems admit linear vertex-kernels whenever they admit constant-factor approximation algorithms (Section 3). Surprisingly, our kernels mainly use a modification of the classical sunflower reduction rule, which usually provides polynomial kernels [4,16,13]. This result implies linear vertex-kernels for \( r \)-Dense Betweenness and \( r \)-Dense Transitive Feedback Arc Set, two natural generalizations of Feedback Arc Set in Tournaments and Dense Betweenness [16]. Both cases were left open by Karpinski and Schudy [16]. Finally, we introduce a different generalization of Feedback Arc Set in Tournaments for constraints of arity \( r \geq 3 \), which allows more freedom on the satisfiability of a constraint. We mainly focus on the case \( r = 3 \). We first state that the problem is \( NP \)-Complete in this case. Next, based on ideas used for Feedback Arc Set in Tournaments [11], we prove that the general case admits a 5-approximation algorithm, and then obtain a linear vertex-kernel (Section 4.3).

\footnote{A tournament is an arbitrary orientation of the complete (undirected) graph.}