Semi-connections and Hierarchies*

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Abstract. Connectivity is the basis of several methodological concepts in mathematical morphology. In graph-based approaches, the notion of connectivity can be derived from the notion of adjacency. In this preliminary work, we investigate the effects of relaxing the symmetry property of adjacency. In particular, we observe the consequences on the induced connected components, that are no longer organised as partitions but as covers, and on the hierarchies that are obtained from such components. These hierarchies can extend data structures such as component-trees and partition-trees, and the associated filtering and segmentation paradigms, leading to improved image processing tools.

Keywords: connectivity, cover hierarchy, connected operators, filtering.

1 Introduction

Connectivity plays a crucial role in the definition of mathematical morphology. Intuitively, the notion of connectivity serves to decide whether a set is either in one piece, or split into several. This notion has been widely studied [2], from axiomatic definitions [21] to variants such as constrained connectivity [24], second-generation connectivity [15,22,7,11], and hyperconnections [14].

Practically, connectivity in discrete image processing is often handled in graph-based frameworks, via the notion of adjacency [16]. In this context, connectivity has led to the development of data structures based on the partition of discrete spaces, and further on partition hierarchies. Examples of such hierarchies are component-trees [19,8], partition-trees [18,24], or hierarchical watershed [49,10]. They can also derive from connectivity hierarchies, leading to partition-trees based, e.g., on fuzzy connectedness [17,1] or second-generation connectivity.

From an applicative point of view, all these concepts have been involved in the development of connected operators [20,23], devoted in particular to image processing tasks such as filtering or segmentation. In this article, we present a preliminary study on the effects of relaxing the symmetry hypothesis, actually required to define adjacency relations (Secs. 2,3). We observe that partitions then become covers, which leads us to define cover hierarchies instead of partition hierarchies. We prove however that such hierarchies can still be handled as (enriched) tree structures (Sec. 4). This framework generalises standard notions such as component-trees or partition-trees (Sec. 5), and provides solutions for performing more accurate antiextensive filtering tasks (Sec. 6).

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2 Background Notions: From Adjacency to Partition Hierarchies

We first recall basic definitions and properties related to the concept of adjacency and some induced notions, namely connectedness, partitions and partition hierarchies.

2.1 Adjacency and Connectedness

Let \( \Omega \) be a nonempty finite set. Let \( \equiv \) be an adjacency (i.e., irreflexive, symmetric, and binary) relation on \( \Omega \). (We recall that \( \equiv \) is a subset of \( \Omega \times \Omega \).) If \( x, y \in \Omega \) satisfy \( x \equiv y \) (and thus \( y \equiv x \)), we say that \( x \) and \( y \) are adjacent.

Let \( X \subseteq \Omega \) be a nonempty subset of \( \Omega \). Let \( \equiv_X \) be the equivalence relation on \( X \) induced by the reflexive-transitive closure of the restriction \( \equiv_X \) of \( \equiv \) to \( X \). If \( x, y \in X \) satisfy \( x \equiv_X y \), we say that \( x \) and \( y \) are connected (in \( X \)). In particular, the equivalence classes of \( X \) associated to the relation \( \equiv_X \) are called the connected components of \( X \), and the set of these connected components is noted \( C_{\equiv_X}[X] \).

Remark 1. These definitions are directly linked to classical notions on graphs, considered for the topological modelling of digital images, as introduced, e.g., in [16]. In particular, \( (\Omega, \equiv) \) and \( (X, \equiv_X) \) are irreflexive (non-directed) graphs.

2.2 Partition Hierarchies

The notion of connected component, associated to (any subsets of) \( (X, \equiv_X) \) is important in image analysis. Indeed, the partition \( C_{\equiv_X}[X] \) associated to images defined on \( X \), can be used for filtering and segmentation purpose, by considering approaches that rely on partition hierarchies in the framework of connected operators [20,23].

In this context, there exist two ways to refine \( (X, \equiv_X) \), to build partition hierarchies. The first way is to work on \( X \), and to define subsets \( Y \subseteq X \), i.e., to progressively constrain the spatial part of \( (X, \equiv_X) \). Practically, defining \( (Y, \equiv_Y) \) such that \( Y \subseteq X \), implies that \( \equiv_Y = (\equiv_X \cap (Y \times Y)) \). The second way is to work on \( \equiv_X \), and to define (symmetric) subrelations \( \equiv_X \subseteq \equiv_X \), inducing connectedness relations \( \equiv_{\leq} \subseteq \equiv_X \), i.e., to progressively constrain the structural part of \( (X, \equiv_X) \).

Remark 2. In the framework of graphs, \( (Y, \equiv_Y) \) is a subgraph of \( (X, \equiv_X) \), and \( (X, \equiv_{\leq}) \) is a partial graph of \( (X, \equiv_X) \).

In both cases, we have the following property.

Property 3. Let \( x \in Y \) (resp. \( x \in X \)). Let \( C = C^x_{\equiv_X}(Y) \in C_{\equiv_X}[Y] \) (resp. \( C = C^x_{\equiv_X}(X) \in C_{\equiv_X}[X] \)), and \( C^x_{\equiv_X}(X) \in C_{\equiv_X}[X] \) be the unique connected components containing \( x \). We have

\[
C \subseteq C^x_{\equiv_X}(X)
\]

(1)

Moreover, for any \( K \in C_{\equiv_X}[X] \), we have

\[
(K \cap C \neq \emptyset) \Rightarrow (K = C^x_{\equiv_X}(X))
\]

(2)