Determinant versus Permanent: Salvation via Generalization?

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Abstract. The fermionant $\mathrm{Ferm}_n^k(\bar{x}) = \sum_{\sigma \in S_n} (-k)^{c(\pi)} \prod_{i=1}^n x_{i,j}$ can be seen as a generalization of both the permanent (for $k = -1$) and the determinant (for $k = 1$). We demonstrate that it is VNP-complete for any rational $k \neq 1$. Furthermore it is $\#P$-complete for the same values of k. The immanant is also a generalization of the permanent (for a Young diagram with a single line) and of the determinant (when the Young diagram is a column). We demonstrate that the immanant of any family of Young diagrams with bounded width and at least n^ϵ boxes at the right of the first column is VNP-complete.

1 Introduction

In algebraic complexity (more specifically Valiant's model[2]) one of the main question is to know whether VP = VNP or not. Answering this is considered to be a very good step towards the resolution of $P = NP$. This question is very close to the question per vs. det, where we ask if the permanent can be computed in polynomial time in the size of the matrix, as is the determinant.

The main idea of this paper is to find a generalization of both the permanent and the determinant in order to study exactly where the difference between them lies. A generalization is here understood as a parameter, let us say t, and a function $f(t, \bar{x})$ such that for example $f(0, \bar{x}) = \det(\bar{x})$ and $f(1, \bar{x}) = \mathrm{per}(\bar{x})$. If we have a complete classification of the complexity of $f(t, \bar{x})$ for any t (with t fixed), we should be able to see where we step from VP to VNP and maybe understand a little bit more why the permanent is hard and not the determinant.

Here we study two different generalizations. First the fermionant, secondly the immanant. The fermionant was introduced by Chandrasekharan and Wiese [3] in 2011 in a context of quantum physics. It is defined with a real parameter k such that for $k = 1$ it is the determinant and for $k = -1$ it is the permanent. Mertens and Moore [7] have demonstrated its hardness for $k \geq 3$ (and with a weaker hardness for $k = 2$), in the framework of counting complexity.

Likewise, but in a different framework and with a complete different proof, we demonstrate the hardness of the fermionant seen as a polynomial for any rational $k \neq 1$ (and of course for $k \neq 0$). This give a interesting point of view on where the hardness of the permanent lies. We also get a bonus: we use a

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technique developed by Valiant to demonstrate the hardness of the fermionant in the counting complexity framework for $k \neq 1$. We thus extend the results of Mertens and Moore [7], in particular to the case $k = 2$, which is, from what I understand, the most interesting case for physicists.

The second generalization is more classical and comes from the field of group representation. It is the immanant, introduced by Littlewood [6] in 1940. Immanants are families of polynomials indexed by Young diagrams. If the Young diagrams are a single column with n boxes, the immanant is the determinant. At the opposite end, if it is a single line of n boxes, the immanant is the permanent. The main question is: for which Young diagrams do we step from VP to VNP?

We know that if there are only a finite number of boxes on the right of the first column, the immanant is still in VP (cf [2]). On the other hand, a few hardness results have been found, fundamentally for Young diagrams in which the permanent is hidden. For example, the hook (a line of n boxes and a column of any number of boxes) and the rectangle (any number of lines each with n boxes) are hard (cf [2]), or more generally if the maximal difference between the size of two consecutive lines is as big as a power of n (cf [1]).

Here we shall demonstrate that for Young diagrams with only two columns, each with n boxes, the immanant is hard, which was an open question (cf [2] Problem 7.1). As each line of these Young diagrams has length no more than two, the permanent is not hidden in there. More generally for any family of Young diagrams with a bounded number of columns and with at least n^ϵ boxes at the right of the first column, the immanant is hard. It has been conjecture that it is still hard if we remove the bounded condition(cf [7] for example).

For a complete classification of the immanant in algebraic complexity, one "just" has to determine the complexity of the ziggurat: the Young diagrams where the first line has n boxes, the second $n - 1$, the third $n - 2$ etc. and the last 1 box. This immanant is most probably also hard. The complexity of the immanant with a logarithmic number of boxes to the right of the first column is also unknown.

2 Definitions

We work within Valiant's algebraic framework. Here is a brief introduction to this complexity theory. For a more complete overview, see [2].

An *arithmetic circuit* over $\mathbb{Q}$ is a labeled directed acyclic connected graph with vertices of indegree 0 or 2 and only one sink. The vertices with indegree 0 are called *input gates* and are labeled with variables or constants from $\mathbb{Q}$. The vertices with indegree 2 are called *computation gates* and are labeled with $\times$ or $+$. The sink of the circuit is called the *output gate*.

The polynomial computed by a gate of an arithmetic circuit is defined by induction: an input gate computes its label; a computation gate computes the product or the sum of its children's values. The polynomial computed by an arithmetic circuit is the polynomial computed by the sink of the circuit.