The Phase Transition in the Erdős–Rényi Random Graph Process

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We shall review the foundation of the theory of random graphs by Paul Erdős and Alfréd Rényi, and sketch some of the later developments concerning the giant component, including some very recent results.

1. Introduction

The theory of random graphs was founded in the late 1950s and early 1960s by the serendipitous partnership of Paul Erdős and Alfréd Rényi. Although they both worked in combinatorics and in probability, Erdős was the quintessential combinatorialist and Rényi the quintessential probabilist: working together, their formidable partnership was ideal for laying the foundations of a cohesive theory of random graphs. In this paper we shall review some of their ground-breaking results together with recent developments concerning the phase transition in graphs and in hypergraphs.

Our paper is organized as follows. In the next three sections we shall present some of the highlights of the work of Erdős and Rényi on the foundation of the theory of random graphs, emphasizing their ground-breaking results on the phase transition, the sudden emergence of the ‘giant component’ as our random graph acquires more and more edges. Section 5 is about the re-awakening of the interest in this phase transition, and the first results on its finer nature, correcting some misconceptions, together with a number of related results. In Section 6 we present some of the recent results proved on the phase transition in the standard random graph process. We do not and cannot aim for a comprehensive account since in the last twenty

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years there has been tremendous activity in the area. We intend to present some of the most important results, but our selection is bound to be greatly influenced by personal preferences. For more detailed accounts of the work of Erdős on probability theory and random graphs, see, e.g., [30] and [33].

In Section 7 we shall turn to models carrying more structure than the Erdős–Rényi graphs. These models include the configuration model for the space of graphs with a given degree sequence, some preferential attachment models like the LCD model (the mathematically precise form of the BA model), the BJR model, encompassing a huge array of earlier models, and the analogue of this model with clustering.

Our presentation is self-contained: all we shall assume is that the reader is familiar with the basic concepts of graph theory and probability. The notation we shall use is standard (see, e.g., [31]) although, when quoting from the papers of Erdős and Rényi, we use their somewhat unusual notation. The results of Erdős and Rényi described in the first part of this paper have of course been presented in many places, for example the books [32, 96] and the survey [108].

2. Erdős and Rényi: The Beginning

“Let us consider a “random graph” \( \Gamma_{n,N} \) having \( n \) possible (labelled) vertices and \( N \) edges; in other words, let us choose at random (with equal probabilities) one of the \( \binom{n}{2} \) possible graphs which can be formed from the \( n \) (labelled) vertices \( P_1, P_2, \ldots, P_n \) by selecting \( N \) edges from the \( \binom{n}{2} \) possible edges \( P_iP_j \) \((1 \leq i < j \leq n)\).”

With this sentence, the very first sentence of [74], Erdős and Rényi launched the theory of random graphs. They start in medias res, as much as that is possible in mathematics, mentioning some earlier related results only later. As a homage to them we shall reproduce the most important results of this paper.

In [74], Erdős and Rényi were interested in the probability that \( \Gamma_{n,N} \) is connected and in the structure of a ‘typical’ graph \( \Gamma_{n,N} \), when \( N \) is in the vicinity of \( N_0 = N_0(n) \) such that \( \Gamma_{n,N_0} \) is connected with probability bounded away from both 0 and 1. Before we state their results, we spell out the definition of \( \Gamma_{n,N} \). Let \( \mathcal{G}_{n,N} \) be the set of all graphs with vertex set \([n] = \{1, \ldots, n\}\) and \( N \) edges, so that the cardinality of this set is

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|\mathcal{G}_{n,N}| = \binom{n}{2} \binom{N}{N}.
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