This chapter presents general features of the dynamics of chains of aligned balls. The dissipation and the dispersion of the kinetic energy at an impact are studied, independently of any impact law. A dispersion index and a dissipation index are defined, which will be used all through the next chapters. Optimization under constraints is used to analyze the variations of these two indices, where the constraints are imposed by the physics (momentum conservation, energetic consistency, kinematic consistency). A 3-ball chain is analyzed in detail.

2.1 Dynamics of a Chain of Aligned Balls

Let us consider a chain of $N$ balls, each of which is constrained to move on a frictionless straight line in order to ensure colinear collisions between the balls, as illustrated in Figure 2.1. Each ball has a radius $R_i$ and is located at position $x_i$. The number of contacts $s$ in the chain equals $N - 1$. The balls in the chain are indexed as 1, 2, ..., $N$ and the contacts are indexed as 1, 2, ..., $N - 1$. Contact $i$ is between balls $i$ and $i + 1$. Initially, the first ball moves with a transitional velocity $V_s$ and strikes the other balls that are at rest and barely touch each other. According to Definition 1.1 this is a multiple impact problem where the struck surface has codimension $N - 1$. Note that, due to central collisions between balls, there is no rotation of the balls during the impacts. A question that arises here is how we can determine the velocities of the balls after impact. Despite the fact that a chain of balls is apparently simple, the answer to the above question is not simple at all. In the following, we will discuss how the multiple impact problem in a chain of balls can be modeled.
2.1.1 Lagrangian Dynamics

Let us describe the state of a chain of balls by a generalized coordinate vector: \( \mathbf{q} = [x_1, x_2, ..., x_N]^T \). This system is subjected to \( N - 1 \) unilateral constraints:

\[
g_i(\mathbf{q}) = x_{i+1} - x_i - (R_{i+1} + R_i) \geq 0, \quad \forall i = 1, 2, ..., N - 1.
\]

These unilateral constraints define the feasible region \( \Phi \):

\[
\Phi = \{ \mathbf{q} \in \mathbb{R}^N | g_i(\mathbf{q}) \geq 0, \quad \forall i = 1, 2, ..., N - 1 \} \tag{2.2}
\]

within which the system has to evolve. The right velocity \( \mathbf{u}^+ = [V_1^+, V_2^+, ..., V_N^+]^T \) is constrained to belong to the convex tangent cone \( T_\Phi(\mathbf{q}) \) to the feasible region \( \Phi \) at point \( \mathbf{q} \):

\[
T_\Phi(\mathbf{q}) = \{ \mathbf{u} \in \mathbb{R}^N | \nabla g_i \mathbf{u} \geq 0, \quad i = 1, 2, ..., N - 1 \}. \tag{2.3}
\]

The dynamics of the chain of balls under consideration is described by the Lagrangian equation and the complementarity condition between the gap function \( g_i(\mathbf{q}) \) and the contact force \( \lambda_i \):

\[
\begin{cases}
M\ddot{\mathbf{q}}(t) = \mathbf{F}_{\text{ext}}(t) + \mathbf{W}\lambda(t) \\
0 \leq g_i(\mathbf{q}) \perp \lambda_i(t) \geq 0, \quad i = 1, 2, ..., N - 1,
\end{cases} \tag{2.4}
\]

where:

- \( \mathbf{M} \) is the mass matrix defined as:

\[
\mathbf{M} = \begin{bmatrix}
m_1 & 0 & \cdots & 0 \\
0 & m_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & m_N
\end{bmatrix}_{N \times N} \tag{2.5}
\]

with \( m_i \) being the mass of ball \( i \);

- \( \mathbf{F}_{\text{ext}} \) is the external force applied to the system;