Expressive Reasoning on Tree Structures: Recursion, Inverse Programs, Presburger Constraints and Nominals

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Abstract. The Semantic Web lays its foundations on the study of graph and tree logics. One of the most expressive graph logics is the fully enriched $\mu$-calculus, which is a modal logic equipped with least and greatest fixed-points, nominals, inverse programs and graded modalities. Although it is well-known that the fully enriched $\mu$-calculus is undecidable, it was recently shown that this logic is decidable when its models are finite trees. In the present work, we study the fully-enriched $\mu$-calculus for trees extended with Presburger constraints. These constraints generalize graded modalities by restricting the number of children nodes with respect to Presburger arithmetic expressions. We show that the logic is decidable in EXPTIME. This is achieved by the introduction of a satisfiability algorithm based on a Fischer-Ladner model construction that is able to handle binary encodings of Presburger constraints.

1 Introduction

The $\mu$-calculus is an expressive propositional modal logic with least and greatest fixed-points, which subsumes many temporal, modal and description logics (DLs), such as the Propositional Dynamic Logic (PDL) and the Computation Tree Logic (CTL) [2]. Due to its expressive power and nice computational properties, the $\mu$-calculus has been extensively used in many areas of computer science, such as program verification, concurrent systems and knowledge representation. In this last domain, the $\mu$-calculus has been particularly useful in the identification of expressive and computationally well-behaved DLs [2], which are now known the standard ontology language OWL for the W3C. Another standard for the W3C is the XPath query language for XML. XPath also takes an important role in many XML technologies, such as XProc, XSLT and XQuery. Due to its capability to express recursive and multi-directional navigation, the $\mu$-calculus has also been successfully used as a framework for the evaluation and reasoning of XPath queries [14]. Since the $\mu$-calculus is as expressive as the monadic second order logic (MSOL), it has been also successfully used in the XML setting in the description of schema languages [1], which can be seen as the tree version of regular expressions. Analogously as regular expressions are interpreted as sets
of strings, XML schemas (regular tree expressions) are interpreted as sets of unranked trees (XML documents). For example, the expression $p(q^*)$ represents the sets of trees rooted at $p$ with either none, one or more children subtrees matching $q$. See figure 1(a) for an interpretation of $p(q^*)$. However, it is well-known that expressing arithmetical constraints goes beyond regularity [1]. For instance, $p(q > r)$ denotes the trees rooted at $p$ with more $q$ children than $r$ children. In figure 1(b) is depicted an interpretation of $p(q > r)$. In the present work, we study an extension of the $\mu$-calculus for trees with Presburger constraints that can be used to express arithmetical restrictions on regular (tree) languages.

**Related Work.** The extension of the $\mu$-calculus with nominals, inverse programs and graded modalities is known as the fully enriched $\mu$-calculus [2]. Nominals are intuitively interpreted as singleton sets, inverse programs are used to express past properties (backward navigation along accessibility relations), and graded modalities express numerical constraints on the number immediate successors nodes [2]. All of them, nominals, inverse programs and graded modalities are present in OWL. However, the fully enriched $\mu$-calculus was proven by Bonatti and Peron to be undecidable [3]. Nevertheless, Bárcañas et al. [1] recently showed that the fully enriched $\mu$-calculus is decidable in single exponential time when its models are finite trees. Graded modalities in the context of trees are used to constrain the number of children nodes with respect to constants. In this work, we introduce a generalization of graded modalities. This generalization considers numerical bounds on children with respect to Presburger arithmetical expressions, as for instance $\phi > \psi$, which restricts the number of children where $\phi$ holds to be strictly greater than the number of children where $\psi$ is true. Other works have previously considered Presburger constraints on tree logics. MSOL with Presburger constraints was shown to be undecidable by Seidl et al. [9]. Demri and Lugiez proved a PSPACE bound on the propositional modal logic with Presburger constraints [5]. A tree logic with a fixed-point and Presburger constraints was shown to be decidable in EXPTIME by Seidl et al. [10]. In the current work, we push further decidability by allowing nominals and inverse programs in addition than Presburger constraints.

**Outline.** We introduce a modal logic for trees with fixed-points, inverse programs, nominals and Presburger constraints in Section 2. Preliminaries of the satisfiability algorithm for the logic are described in Section 3. In Section 4 an EXPTIME satisfiability algorithm is described and proven correct. A summary together with a discussion of further research is reported in Section 5.