Fourier Representation of Sequences

The Fourier representation of analog signals was discussed in the previous chapter. This representation is now extended to data sequences, and digital signals. To this end, the discrete Fourier transform (DFT) is defined and several of its properties are developed. Specifically, the convolution and correlation theorems are described and the spectral properties such as amplitude, phase, and power spectra are developed. By illustrating the 2-dimensional DFT, it is shown that the DFT can be extended to multiple dimensions. Finally, the concepts of time-varying Fourier power and phase spectra are introduced.

3.1 Definition of the Discrete Fourier Transform

If \( \{X(m)\} \) denotes a sequence \( X(m), m = 0, 1, \ldots, N - 1 \) of \( N \) finite valued real or complex numbers, then its discrete [1] or finite [2] Fourier transform is defined as

\[
C_x(k) = \frac{1}{N} \sum_{m=0}^{N-1} X(m)W^{km}, \quad k = 0, 1, \ldots, N - 1 \tag{3.1-1}
\]

where \( W = e^{-j2\pi/N}, j = \sqrt{-1} \). The exponential functions \( W^km \) in Eq. (1) are orthogonal such that

\[
\sum_{m=0}^{N-1} W^{km}W^{-lm} = \begin{cases} N, & \text{if } (k - l) \text{ is zero or an integer multiple of } N \\ 0, & \text{otherwise} \end{cases} \tag{3.1-2}
\]

Now, from Eq. (1) we have

\[
\sum_{k=0}^{N-1} C_x(k) W^{-km} = \frac{1}{N} \sum_{k=0}^{N-1} W^{-km} [X(0) + X(1)W^k + \cdots + X(m)W^{km} + \cdots + X(N-1)W^{k(N-1)}] \tag{3.1-3}
\]

Application of Eq. (2) to Eq. (3) results in the inverse discrete Fourier transform (IDFT) which is defined as

\[
X(m) = \sum_{k=0}^{N-1} C_x(k) W^{-km}, \quad m = 0, 1, \ldots, N - 1 \tag{3.1-4}
\]
Since Eqs. (1) and (4) constitute a transform pair, it follows that the representation of the data sequence \( \{X(m)\} \) in terms of the exponential functions \( W^{km} \) is unique.

We observe that functions \( W^{km} \) are \( N \)-periodic; that is,

\[
W^{km} = W^{(k+N)m} = W^{k(m+N)}, \quad k, m = 0, \pm 1, \pm 2, \ldots
\]

Consequently the sequences \( \{C_{\pi}(k)\} \) and \( \{X(m)\} \) as defined by Eqs. (1) and (4) are also \( N \)-periodic. That is, the sequences \( \{X(m)\} \) and \( \{C_{\pi}(k)\} \) satisfy the following conditions:

\[
X(\pm m) = X(sN \pm m) \quad s = 0, \pm 1, \pm 2, \ldots
\]

\[
C_{\pi}(\pm k) = C_{\pi}(sN \pm k), \quad s = 0, \pm 1, \pm 2, \ldots
\]

Using Eqs. (5) and (6) it can be shown that

\[
\sum_{m=p}^{q} X(m) W^{km} = \sum_{m=0}^{N-1} X(m) W^{km}
\]

(3.1-7)

and

\[
\sum_{k=p}^{q} C_{\pi}(k) W^{-km} = \sum_{k=0}^{N-1} C_{\pi}(k) W^{-km}
\]

(3.1-8)

when \( p \) and \( q \) are such that \( |p - q| = N - 1 \).

It is generally convenient to adopt the convention [2]

\[
X(m) \leftrightarrow C_{\pi}(k)
\]

to represent the transform pair defined by Eqs. (1) and (4).

### 3.2 Properties of the DFT

A detailed discussion of the properties of the DFT is available in [2, 4]. In what follows we consider a few of these properties which are of interest to us.

1. **Linearity Theorem.** The DFT is a linear transform; i.e., if

\[
X(m) \leftrightarrow C_{\pi}(k)
\]

and

\[
Z(m) = aX(m) + bY(m)
\]

then

\[
C_{\pi}(k) = aC_{\pi}(k) + bC_{\gamma}(k)
\]

(3.2-1)

2. **Complex Conjugate Theorem.** If \( \{X(m)\} = \{X(0) \ X(1) \ldots X(N - 1)\} \) is a real-valued sequence such that \( N/2 \) is an integer, and

\[
X(m) \leftrightarrow C_{\pi}(k)
\]