Abstract. The growth of non-saturating and saturating populations is modeled by a general kind of stochastic differential equation. The transition density functions of the solutions of these equations, obtained using the Stratonovich stochastic integral, are obtained in closed form. Moments, first passage time probability densities and probabilities of extinction can be found explicitly in a number of cases. Specifically considered are Malthusian growth, a general non-saturating process, a general saturating process which contains the Pearl-Verhulst model as a special case, and Gompertzian growth. This last-named process is examined with a view to the stochastic modeling of large populations of tumor cells.

1. Introduction

Recently there has been much interest in problems connected with the effects of random influences on the growth of populations. Goel et al. (1971) and Montroll (1972) have studied populations whose size $N(t)$ at time $t$ satisfies a stochastic differential equation of the form

$$dN(t) = kN(t)G(N(t)/K)\,dt + N(t)\,dw(t), \quad (1)$$

where $G(\cdot)$ is a saturation function, $K$ is the (constant) saturation level, $k$ is a growth rate parameter and $w(t)$ is a Wiener process with zero mean. The random term $N(t)\,dw(t)$ might reflect fluctuations in the physical features of the environment or the effects of other species. In the latter case, $n$ equations of type (1) can be used as a (degenerate) model of the Lotka-Volterra type where the number of species is $n$.

There seems, however, to be a basic problem in the approach of the above-mentioned references in that when the random term is added to the deterministic equation, which represents a saturating (bounded) process, the resulting random process $N(t)$ takes values in $[0, \infty)$. Hence the
"saturating" feature is, in the strict sense, lost.

In this paper we will consider populations whose sizes satisfy stochastic equations of the type

$$dN(t) = h(N(t))dw(t),$$

where the function $h(N)$ may or may not be of the saturating form. Such equations imply that a growth rate parameter will be taken to be a Gaussian white noise, which may be looked upon as the formal derivative, $dw(t)/dt$, of a Wiener process. Furthermore, we will not necessarily assume that $w(t)$ has a zero mean.

It is known that, provided $h(N)$ satisfies certain regularity conditions (Doob, 1953), equation (2) has the solution

$$N(t) = N(0) + \int_0^t h(N(t'))dw(t'),$$

which is a continuous Markov process. That is, $N(t)$ is a diffusion process whose transition probability density function (p.d.f.), satisfies the Fokker-Planck equation

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial N} [M(N)p] + \frac{V}{2} \frac{\partial^2}{\partial N^2} [h(N)^2p],$$

where

$$p(N,t|N_0)dN = \Pr[N < N(t) \leq N+dN|N(0) = N_0],$$

$V$ being the variance parameter of $w(t)$.

Ito and Stratonovich Calculi.

The first infinitesimal moment, $M(N)$, depends upon the way in which the stochastic integral in (3) is defined. Using obvious subscripts to denote whether the Ito or Stratonovich definition is being employed (Jaswinski, 1970), we have

$$M_I(N) = mh(N),$$
$$M_S(N) = mh(N) + \frac{Vh(N)h'(N)}{2},$$

where $m$ is the mean value of $dw(t)/dt$ and the prime denotes differentiation. The additional term in the Stratonovich first moment indicat-