Nonexistence of Periodic Solutions for a Class of Epidemiological Models

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*Dedicated, on the occasion of his 65th birthday, to Kenneth L. Cooke who inspired us both to work in this area.*

**Abstract**

Disease transmission models are formulated under assumptions that the size of the population varies and the force of infection is of the proportionate mixing type. Conditions are given that rule out the possibility of periodic solutions for such models. Examples are considered and sharp thresholds identified.

**1 Introduction**

The behavior of a general *SAIS* (*S* = susceptible, *A* = asymptomatic, *I* = infective,) epidemiological model in a population of varying size is governed by a differential equation in $\mathbb{R}_+^3$. When the force of infection is of the proportionate mixing type, the nonlinear terms of this equation are homogeneous of degree one and satisfy a balance condition. Moreover, in the linear part, the off-diagonal terms are nonnegative.

We consider a general equation of this form, and give conditions which rule out periodic solutions, including limit cycles, homoclinic orbits and oriented phase polygons. Our method involves a new technique and extends results of Busenberg and van den Driessche (1990) for an *SIRS* (*R* = recovered) model, in which a generalization of the Bendixson-Dulac criterion is proved and used in the analysis.

In certain special cases, the nonexistence of periodic solutions, in combination with analysis of the existence and stability of equilibrium points, provides a complete global analysis of the model.
2 Mathematical formulation

We consider a differential equation in $\mathbb{R}^3_+$ of the form

$$x' = Mx + f(x)$$

(2.1)

where $'$ denotes the derivative $d/dt$. The $3 \times 3$ constant matrix $M = [m_{ij}]$ is assumed to be essentially nonnegative, that is the off-diagonal entries of $M$ are nonnegative, and it is also assumed that at least one is positive. The nonlinear function $f : \mathbb{R}^3_+ \rightarrow \mathbb{R}^3_+$ is assumed to be continuously differentiable and homogeneous of degree one, that is $f(ax) = af(x)$ for $a > 0$, and to obey the balance condition $\sum_{i=1}^{3} f_i = 0$. In the disease transmission models, we are concerned with solutions having nonnegative components $x_i(t) \geq 0$, and we also assume that $(Mx)_i + f_i(x) \geq 0$ when evaluated on $x_i = 0$. For $\sum_{i=1}^{3} x_i \neq 0$, we introduce the normalized variable (see, e.g., Hahn (1967, Section 57), Hadeler et al. (1988), Hofbauer and Sigmund (1988), Busenberg and Hadeler (1990), Busenberg and van den Driessche (1990))

$$y = x / \sum_{i=1}^{3} x_i,$$

(2.2)

with $\sum_{i=1}^{3} y_i = 1$. The normalized variable satisfies

$$y' = My - (1 \cdot My)y + f(y),$$

(2.3)

where 1 denotes the vector in $\mathbb{R}^3$ with every entry equal to one. Clearly, the hyperplane $S \equiv \left\{ y : \sum_{i=1}^{3} y_i = 1 \right\}$ is invariant under the flow induced by (2.3), since

$$\left( \sum_{i=1}^{3} y_i \right)' = (1 \cdot My) \left( 1 - \sum_{i=1}^{3} y_i \right) + \sum_{i=1}^{3} f_i(y) = 0.$$

We now write (2.3) in component form for $i = 1, 2, 3$:

$$y_i = (My)_i - (1 \cdot My)y_i + f_i(y_1, y_2, y_3).$$

(2.4)

On $S$ we can write the first equation of (2.4) in the following alternate forms:

$$y_1' = f_{12}(y_1, y_2) + f_1(y_1, y_2, 1-y_1-y_2) = f_{13}(y_1, y_3) + f_1(y_1, 1-y_1-y_3, y_3),$$

(2.5)

where