CHAPTER 3

Local Properties of Distributions

Topology of open and closed sets in Euclidean space; coverings; Bolzano–Weierstrass and Heine–Borel theorems; shrinkage principle; partitions of unity; comparison of distributions in an arbitrary open set; piecing-together principle; support of a distribution; derivative as a local property.

Prerequisite: Chapter 2

Although a distribution does not have a definite value at a specified value $x$ of its argument, one can discuss its properties in any arbitrarily small neighborhood of $x$. Such local properties are discussed in this chapter.

3.1 Quick Review of Open and Closed Sets in $\mathbb{R}^n$

A point $x$ in a set $S$ is called an interior point of $S$ if, for some sufficiently small $\varepsilon > 0$, the ball $B = \{y: \|y - x\| < \varepsilon\}$ with center at $x$ and radius $\varepsilon$ lies in $S$ (i.e., if every point $y$ of the ball $B$ is a point of $S$). A set $S$ is called open if every point of $S$ is an interior point. For example, the ball $B$ itself is an open set, because if $y$ is any point of $B$, and $\varepsilon' = \varepsilon - \|y - x\|$, then the ball $B' = \{z: \|z - y\| < \varepsilon'\}$ lies in $B$. See Figure 3-1.

A set is called closed if it contains all its limit points. The closed ball $B = \{y: \|y - x\| \leq \varepsilon\}$ is a closed set (the points on the surface have now been included). A set $S$ is closed if its complement $\mathbb{R}^n - S$ is open, and conversely. A set $S$ together with all its limit points is a closed set, called the closure of $S$, and is denoted by $\bar{S}$. If $S$ itself is closed, then $\bar{S} = S$.

Exercises

1. Show that if $f(x)$ is a continuous real function on $\mathbb{R}^n$, then the sets

   \[ \{x: f(x) > 0\}, \quad \{x: f(x) \neq 0\}, \quad \{x: a < f(x) < b\} \]
are open sets, while the sets
\[ \{x: f(x) \geq 0\}, \quad \{x: f(x) = 0\}, \quad \{x: a \leq f(x) \leq b\} \]
are closed.

The union of an arbitrary collection of open sets is open, and so is the intersection of a finite collection of open sets. In the corresponding statements about closed sets, the words "arbitrary" and "finite" have to be interchanged.

2. Prove the foregoing statements and discuss the unions and intersections of the following collections of intervals on \( \mathbb{R} \) (in each case, \( k = 1, 2, \ldots \)): (1) \( |x| < 1 - (1/k) \), (2) \( |x| \leq 1 - (1/k) \), (3) \( |x| < 1 + (1/k) \), (4) \( |x| \leq 1 + (1/k) \).

For any function \( f \), the closure of the set \( \{x: f(x) \neq 0\} \) is called the support of \( f \).

According to the Bolzano–Weierstrass theorem, any sequence \( \{x_j\}_1^\infty \) in a bounded set \( \mathcal{S} \) in \( \mathbb{R}^n \) has a convergent subsequence; if \( \mathcal{S} \) is also closed, the limit of the subsequence lies in \( \mathcal{S} \).

For any set \( \mathcal{S} \), if there is given a collection of open sets \( \{\Omega, \Omega', \Omega'', \ldots\} \) — possibly an infinite or even uncountable collection — such that every point \( x \) of \( \mathcal{S} \) lies in at least one of them, then the collection is called an open covering of \( \mathcal{S} \). According to the Heine–Borel theorem, if furthermore \( \mathcal{S} \) is a closed bounded set in \( \mathbb{R}^n \), then there is a finite subcollection, which will be called \( \{\Omega_i: i = 1, \ldots, N\} \), of the above collection that also covers \( \mathcal{S} \); that is, every point of \( \mathcal{S} \) lies in at least one of the sets \( \Omega_i(i = 1, \ldots, N) \). (The number \( N \) generally depends, for a given \( \mathcal{S} \), on the particular open covering in question.) See Natanson (1955), where, however, the theorem is called the Borel covering theorem.

[In any topological space, a set \( K \) is called compact if it has the above property that every open covering of \( K \) contains a finite covering of \( K \), and it is called sequentially compact if every sequence lying in \( K \) has a subsequence that converges to a limit in \( K \). In any metric space, the two concepts are equivalent. In \( \mathbb{R}^n \), a set is compact if and only if it is closed and bounded.]

**Lemma 1.** If \( K \) is a closed bounded set in \( \mathbb{R}^n \) contained in an open set \( \Omega \), then the distance \( d \) from \( K \) to the complement of \( \Omega \) is positive. I.e., there is a margin around \( K \) in \( \Omega \), whose width is nowhere less than \( d \).

**Proof.** The distance \( d \) is given by
\[ d = \inf \{\|x - y\|: x \in K, y \notin \Omega\}. \quad (3.1-1) \]