CHAPTER 6

Some Problems Connected with the Laplacian

Vibration eigenfunctions in a bounded domain; variational methods; the Dirichlet integral; the potential due to a given charge distribution; Poisson's equation; convolutions; the direct product; Schwartz's nuclear theorem; the Cauchy-Riemann equations; harmonic functions.

Prerequisite: Chapter 5

The Laplacian is in many respects of a more classical nature than many of the differential operators to be discussed in Chapters 10 and 11. One of the basic problems is to find the eigenfunctions $u(x)$ of the equation $\nabla^2 u + \lambda u = 0$ in a region $\Omega$ of $n$-dimensional space with the boundary condition $u(x) = 0$ on the boundary $\partial \Omega$. For $n = 2$ that is the classical problem of a vibrating membrane. More generally, for both $n = 2$ and $n = 3$ the eigenfunctions and the variational methods that determine them are useful in problems of vibration, heat flow, electromagnetic fields, and hydrodynamic stability. That is the main subject of this chapter.

Some additional topics illustrate the universality of distribution-theory methods. First, the validity of Poisson's equation for the potential $V(x)$ due to a charge with density $\rho(x)$ is established, when $\rho(x)$ is an arbitrary distribution with bounded support in $\mathbb{R}^3$. Second, it is shown that the Cauchy-Riemann equations for distributions $u$ and $v$ on $\mathbb{R}^2$ imply the analyticity of $u + iv$ more generally than in classical theory, if the derivatives in the Cauchy-Riemann equations are interpreted in the distribution theory sense. In that connection it is shown that any harmonic distribution in $\mathbb{R}^n$ is a harmonic function in $\mathbb{R}^n$.

Convolutions of distributions are discussed briefly, because they are needed in the discussion of Poisson's equation, and then Schwartz's Nuclear Theorem is discussed because it is needed for a full understanding of convolutions.
6.1 The Potential; Poisson's Equation

It is recalled that in electrostatics the potential $V(x)$ due to a distributed charge with density $\rho(x)$ is given by

$$V(x) = \int_{\mathbb{R}^3} \frac{1}{|x - y|} \rho(y) d^3y$$

(6.1-1)

and that this potential satisfies Poisson's equation

$$\nabla^2 V = -4\pi \rho.$$  \hspace{1cm} (6.1-2)

(Single vertical bars are used in this chapter to denote the length of a vector, to permit use of the double bars to denote the $L^2$ norm of a vector field when regarded as an element of a Banach or Hilbert space.)

These equations will be generalized, in the next two sections, to the case in which $\rho$ is any distribution of bounded support on $\mathbb{R}^3$. We discuss briefly also the modified problem in which the charge is contained in a region $\Omega$ bounded by a simple closed surface $\partial \Omega$ on which $V(x) = 0$. Then the first factor in the integrand of (6.1-1) is replaced by a Green's function $G(x, y)$.

6.2 Convolutions

According to (6.1-1), $V(x)$ is the three-dimensional convolution of the functions $1/|x|$ and $\rho(x)$, hence the first problem is to define the convolution $f \ast g$ of two distributions $f$ and $g$ on $\mathbb{R}^n$. If $f$ and $g$ are ordinary functions and $g$ has bounded support, then the convolution is also a function $(f \ast g)(x)$; as a distribution, it is given by

$$\langle f \ast g, \phi \rangle = \int \int f(x - y)g(y)d^n y \phi(x)d^n x \hspace{1cm} (6.2-1)$$

and we shall proceed by direct imitation of this formula. If $g$ is a distribution with bounded support, then the inner integration in the last member of (6.2-1) (the integration with respect to $y$) is to be interpreted as

$$(6.2-2) \hspace{1cm} \langle g, \phi \rangle = \int \int f(w)g(y)\phi(y + w)d^n y d^n w,$$

where

$$\varphi_w(y) = \varphi(w + y). \hspace{1cm} (6.2-3)$$

According to Exercise 3 in Section 2.6 and the accompanying discussion of mollifiers, $\langle g, \varphi_w \rangle$ is a $C^\infty$ function of $w$. It is clear from (6.2-3) that if $|w|$ is large enough, the supports of $g$ and $\varphi_w$ do not overlap, and $\langle g, \varphi_w \rangle$ is zero; that is, the function $\langle g, \varphi_w \rangle$ has bounded support, hence is a test function. The outer integration in (6.2-1) can be interpreted as the result of putting that