A Mathematical Model of the Semicircular Canals

by

William C. Van Buskirk

Department of Biomedical Engineering
Tulane University
New Orleans, Louisiana 70118

1. Introduction

The semicircular canals are the primary transducer for the sensing of angular motion. As such, they are part of the organs of equilibrium. The importance of these organs for the successful functioning of the human body is obvious. The semicircular canals have, therefore, attracted the attention of physiologists, sensory psychologists and physicians over the years. From the very beginning, physical scientists and mathematicians have been consulted to provide an explanation of the mechanics of this fascinating organ.

The advent of aerospace flight with its new demands on the human organism has accelerated the pace of vestibular research. It has become apparent that, while the semicircular canals are an engineering system of some elegance, they are capable of producing disorienting sensations when subjected to non-physiological motion. This knowledge has helped aerospace planners to avoid situations which might prove discomforting or disabling to the pilot or astronaut.

This alone is an adequate reason for wishing to develop a mathematical model of the semicircular canals, but a further incentive comes from the fact that a full understanding of the mechanics of healthy semicircular canals may contribute to the diagnosis and treatment of canals in a diseased state.

2. Anatomy and physiology

The semicircular canals are located, along with the organ of hearing, in the inner ear. There are three sets of canals on each side of the head (see Figure 1). They are oriented in almost mutually orthogonal planes so that rotation about any axis may be properly sensed. As shown, each canal consists of two parts: an outer canal, which is a channel carved in bone, and an inner, membranous canal. The inner canal is filled with a fluid called endolymph. The space between the membranous and bony canal is filled with perilymph, a fluid different in composition from endolymph.

One end of each semicircular duct is enlarged to form its ampulla. The ampulla nearly fills the cross-section of the bony canal and terminates on the utricle. The ampulla contains the cupula, a gelatinous dividing partition with the same density as endolymph. The cupula fills the entire cross-section of the ampulla, thus interrupting the otherwise continuous fluid path through the duct, utricle and ampulla.

The cupula is the system transducer. It is connected to nervous tissue at its base. Mechanical deflection of the cupula is converted into electrical impulses which transmit the state of angular motion along the vestibular nerve to the central nervous system.

Qualitatively, the manner in which the semicircular canals work is as follows. An angular acceleration of the head causes the bony canals and the membranous structure attached to them to accelerate in a similar manner. The inertia of the endolymph, however, causes it to lag behind the motion of the head. Thus there is a flow of endolymph relative to the duct walls. This flow deflects the cupula, initiating the electrical impulses to the brain.

3. Formulation of the problem

In this section, we develop a mathematical model for fluid flow in a single semicircular canal. The membranous semicircular canal duct is approximated by a section of a rigid torus filled with an incompressible Newtonian fluid. For the
purpose of this analysis, the perilymph is assumed to have no effect on the deflection of the cupula. The governing equation for the flow of fluid in the duct is the classical Navier-Stokes equation

$$\frac{\partial \tilde{v}}{\partial t} + \tilde{v} \cdot \nabla \tilde{v} = -\frac{1}{\rho} \nabla p + \tilde{b} + \nabla \tilde{\gamma} \tilde{v}$$

(1)

where $\tilde{v}$ is the velocity of the fluid with respect to an inertial reference frame, $\rho$ is the density, $p$ is the pressure, $\tilde{b}$ is the body force and $\gamma$ is the kinematic viscosity.

We are interested in the flow of the fluid with respect to the duct. Therefore we introduce the symbol $u$, which will represent the velocity of the fluid relative to the duct wall. If $\tilde{v}_w$ is the velocity of the wall, then

$$\tilde{v} = u + \tilde{v}_w.$$  

(2)

Now the velocity of a given point on the duct wall is given by

$$\tilde{v}_w = \tilde{v}_c + \omega \times (\tilde{R} + \tilde{r}),$$

(3)

where $\tilde{v}_c$ is the velocity of the center of curvature of the duct, $\omega$ is the angular velocity of the canal, $\omega$ is the position vector of the center of curvature, and $\tilde{R}$ is the position vector of the point on the duct wall with respect to the center of the duct (see Figure 2). Since $|\tilde{r}| / |\tilde{R}| \ll 1$, we can approximate $\tilde{v}_w$ by

$$\tilde{v}_w \approx \tilde{v}_c + \tilde{\omega} \times \tilde{R}.$$  

(4)

Therefore

$$\tilde{v} = u + \tilde{v}_c + \tilde{\omega} \times \tilde{R}.$$  

(5)