Triangular Embeddings of Tensor Products of Graphs

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ABSTRACT

Given triangular embeddings of graphs $G$ and $H$ and an assignment of signs $\pm 1$ to the angles of the embedding of $G$, with the property that the sum of the signs around each vertex of $G$ is $\pm 1$ and the product of the four signs at each edge is $+1$, triangular embeddings of the tensor (also called categorical) product $G \otimes H$ of $G$ and $H$ are constructed. This generalizes previously known results about embeddings of tensor products of graphs.

1. INTRODUCTION

Our graphs will be finite and simple. By a surface we mean a compact surface without boundary. We assume that the reader is familiar with basic notions of topological graph theory, and refer to the book [GT].

Let $G$ and $H$ be graphs and let $G \cdot H$ denote some graph product of $G$ and $H$. It is an important problem to determine any genus embeddings of $G \cdot H$, i.e., embeddings of the graph with the minimal possible genus, knowing some genus embeddings of $G$ and $H$.

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In the case of the Cartesian product \( \times \) of graphs, several results about embeddings of the graph \( G \times H \) are known [P1, P2, W]. Cf. also [GT]. There are also some works about the embeddings of the lexicographic [B3] and the tensor products [B4, Z] of graphs. In this paper we consider the tensor (categorical) product.

If \( G \) and \( H \) are graphs then the tensor product of \( G \) and \( H \) is the graph \( G \otimes H \) having vertex set \( V(G \otimes H) = V(G) \times V(H) \) and with vertices \((u, v)\) and \((u', v')\) being adjacent if and only if \( u \) is adjacent to \( u' \) in \( G \) and \( v \) is adjacent to \( v' \) in \( H \). By some authors this product is also called conjunction, or the categorical product.

We shall restrict ourselves to a particular, simpler but still difficult problem: Let \( G \) and \( H \) be given graphs together with triangular embeddings \( i : G \to \Sigma_1 \) and \( j : H \to \Sigma_2 \). Recall that "triangular" means that all the faces of the embeddings are triangles. The problem is to find a triangular embedding of the tensor product \( G \otimes H \). Notice that triangular embeddings of simple graphs are always minimal genus embeddings (either orientable, or nonorientable). It will be assumed, moreover, that all the vertices of the graph \( G \) have odd degrees. For our problem this is a very natural condition. Namely, in order for the graph \( G \otimes H \) to have triangular embeddings, the neighbors of each vertex must span a cycle. The neighborhood graph of a vertex \((u, v)\) in \( G \otimes H \) is equal to the tensor product of neighborhood graphs of \( v \) and \( u \) in \( G \) and \( H \), respectively. If these are both even cycles, for example, their product is disconnected, and hence \( G \otimes H \) trivially admits no triangular embeddings.

We construct triangular embeddings of \( G \otimes H \), \( \kappa : G \otimes H \to \Sigma \), with the property that there are (simplicial) maps \( \pi_1 : \Sigma \to \Sigma_1 \) and \( \pi_2 : \Sigma \to \Sigma_2 \) such that each face in \( \Sigma \) is projected bijectively onto a face in \( \Sigma_i \), \( i = 1, 2 \). Mappings \( \pi_i \) with this property are called coverings with folds (cf. [B1, B2] for more details about such mappings). Combinatorially they define simplicial maps between simplicial complexes corresponding to the triangular embeddings. In our construction, the mapping \( \pi_1 \) will have no folds, so topologically it will be a branched covering of surfaces with branch points at vertices of \( G \) only. Our construction uses certain valuation of the angles of the embedding of the graph \( G \). We believe that such valuations always exist. Our main result, Theorem 1, generalizes previous results [B4] on triangular embeddings and the genus of tensor products of graphs.