The Outerthickness & Outercoarseness of Graphs
I. The Complete Graph & The $n$-Cube

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Many of the results of this paper were announced without proof in [9]; a parallel paper, discussing the complete bipartite graph, will appear in [11]. Some results were also obtained independently by L.S. Mel’nikov and presented at the 6th Hungarian Combinatorics Colloquium in Eger on 81-07-10.

A graph is planar if it can be imbedded in the plane (or sphere). The complement of the graph in such an imbedding is a collection of open cells. If there is an imbedding with all the vertices of the graph on the boundary of a single cell, the graph is said to be outerplanar.

Halin [12, and see also 7] has shown that a graph is outerplanar just if it does not contain a subgraph homeomorphic to the complete graph, $K_4$, or to the complete bipartite graph, $K_{2,3}$.

The thickness of a graph is the least number of planar graphs having the original graph for their union; i.e. the least number of parts in an edge-partition of the graph, with the edges in each part forming a planar graph.

The outerthickness $\theta_o(G)$, of a graph $G$ is defined similarly, but with "outerplanar" in place of "planar".

The outerthickness of the complete graph

It can be seen, from Euler’s formula, for example, that an outerplanar graph on $n$ vertices contains at most $2n-3$ edges. The complete graph, $K_n$, has each of its vertices joined to every other, and contains $\frac{1}{2}n(n-1)$ edges. The outerthickness of the complete graph is therefore at least

$$\left\lfloor \frac{n(n-1)}{2(2n-3)} \right\rfloor = \left\lfloor \frac{n}{4} + \frac{n}{4(2n-3)} \right\rfloor = \left\lfloor \frac{n+1}{4} \right\rfloor \text{ for } n > 1.$$ 

Theorem 1  The outerthickness of $K_n$ is $\theta_o(K_n) = \left\lfloor \frac{n+1}{4} \right\rfloor$, except that $\theta_o(K_7) = 3$.

Proof: From the preceeding remark, the outerthickness is at least $\lceil (n+1)/4 \rceil$. Figure 1 displays equality for $n \leq 6$. 

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To see that $\theta_o(K_7) = 3$ and not 2, note that $K_7$ can be partitioned into three Hamilton circuits (Figure 2), but it is not the union of two outerplanar graphs. An outerplanar graph on 7 vertices contains at most $2 \times 7 - 3 = 11$ edges, so one of the two would contain 11 edges (with 10 in the other). An outerplanar graph with 7 vertices and 11 edges is isomorphic to one of the four graphs in Figure 3 (compare Fig. 11.9 on p. 108 of [13]). In each case the complementary graph contains the non-outerplanar graph $K_{2,3}$ (vertices 1 & 2 each joined to 4, 5 & 6).