The purpose of this communication is to provide a first insight into the problem of getting a recursive structure for two-dimensional filters via the algebraic realization theory. The line undertaken here has several points of contact with the algebraic realization theory of bilinear maps [1, 2].

1. External representation and Nerode equivalence classes

The external representation of a two-dimensional filter is defined as:

\[ S \triangleq (T_1 \times T_2, U, U, Y, Y, F) \]

where:
- \( T_1 = T_2 = \mathbb{Z} \)
- \( U = Y = K \) arbitrary field
- \( U, Y \) are sets of generalized formal power series in two variables over \( K \):
  \[ r = \sum_{-k = i, j}^{\infty} (r, z_1^i z_2^j) z_1^i z_2^j, \quad \text{for some integer } k. \]
- \( F: U \rightarrow Y \) (input-output map): it satisfies:
  (i) linearity
  (ii) two-dimensional shift invariance:
  \[ F(z_1^i z_2^j r) = z_1^i z_2^j F(r), \quad \forall i, j \in \mathbb{Z} \]
  (iii) two-dimensional proper causality:

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implies:

\[(F_{u_1}, z_{1z_2}^{i,j}) = (F_{u_2}, z_{1z_2}^{i,j}), \quad i \leq t_1, \quad j \leq t_2, \quad \forall u_1, u_2 \in U.\]

Under these assumptions it is easy to verify that:

\[s \triangleq F(u) \in (z_1z_2)^K[[z_1, z_2]]\]

and

\[F(u) = su, \quad \forall u \in U.\]

So doing the two-dimensional filters (in their input-output representation) are in one-to-one correspondence with the formal series \((z_1z_2)^K[[z_1, z_2]]\) and vice-versa.

The state introduction via the Nerode equivalence classes represents the way of obtaining a recursive filter in the system theoretic sense. This requires to endow the input space with the structural properties:

(i) **Truncation.** Let

\[r = \sum_{i,j} (r, z_{1z_2}^{i,j}) z_{1z_2}^{i,j}, \quad \forall r \in U;\]

the truncation operator \(T(t_1, t_2): U \to U\) is defined by:

\[T(t_1, t_2)r = \sum_{i \leq t_1, j \leq t_2} (r, z_{1z_2}^{i,j}) z_{1z_2}^{i,j}\]

Let \(U^* \triangleq \{T(0, 0)u: u \in U\}\). Then the map

\[f: U^* \to (z_1z_2)^K[[z_1, z_2]]\]

defined by the assignment:

\[f(u) = \sum_{i,j > 0} (F_{u}, z_{1z_2}^{i,j}) z_{1z_2}^{i,j}\]

characterizes \(S\) in the same sense as \(F\) does. This follows from two-dimensional