8 / Applied Mathematics for Amateur Astronomers

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8.1. Introduction

How greatly astronomy depends on numerical calculations and mathematical formulae is shown by the fact that until late into the nineteenth century astronomy was considered to be part of mathematics. Equally today, the treatment of astronomical problems is unthinkable without mathematics, so that even the amateur astronomer frequently requires a certain minimum of mathematical knowledge. No proofs or derivations of the formulae will be given in this chapter. The reader interested in this aspect can find proofs in books of a more technical nature. Furthermore, it will be assumed that the reader is familiar with logarithms and trigonometric functions, and occasionally we assume acquaintance with the basic notions of differential calculus.

As to the accuracy of numerical calculations it is best to remember the rule of thumb which states that in five-digit calculations the final result can be expected to be accurate to about two or three seconds of arc. Thus, if we only require an accuracy of one minute of arc we only need work in four digits.

8.2. Theory of Errors

8.2.1. Direct Observations

Every measurement is subject to errors because our instruments and our senses are imperfect. We distinguish between two types of errors, systematic and accidental. Systematic errors depend in a known manner on some external circumstances and can be determined, although often only by tedious investigation. Accidental errors, on the other hand, act sometimes in one and sometimes in another direction and by their very nature cannot be predicted. It is only the latter with which the general theory of errors is concerned. Its task is to establish the law of frequency of errors and to judge the accuracy of a measurement or of a calculation.

The fundamental principle of the theory of errors was established by Gauss, and states that the true value of a required quantity for which we
possess a series of measurements has such a value that the sum of the squares of the individual errors is a minimum. This theory does not express a law of nature but rather a definition, although a very plausible one.

In the simplest case we may have a single quantity that can be measured directly. Let there be \( n \) measured values \( l_1, l_2, \ldots, l_n \), which would be identical if there were no errors, but of course this is never true in reality. Here the theory of errors asserts, in accordance with the principle of least squares, that the “most probable value” of the required quantity is given by the arithmetic mean

\[
L = \frac{1}{n} (l_1 + l_2 + \cdots + l_n).
\]

The “mean error” is expressable in terms of the differences \( v_i = l_i - L \) between the individual measurements and the mean value \( L \). Using the customary notation we denote the sum of the squares of the errors by \([vv]\). Then the mean error of a single measurement is

\[
\mu = \sqrt{\frac{[vv]}{n-1}},
\]

and the mean error of the mean is

\[
\mu_L = \frac{\mu}{\sqrt{n}} = \sqrt{\frac{[vv]}{n(n-1)}}.
\]

Thus, by forming the mean of \( n \) individual measurements, the accuracy of the results can be improved by a factor of \( \sqrt{n} \).

These formulae assume that all measurements are equally reliable. In many cases this is not true. For example, an observer with a small telescope of low magnification will be able to measure the distance between the components of a double star less accurately than somebody with a large instrument. Such differences are taken care of by assigning to each measurement a “weight.” The larger the degree of reliability, the larger the weight. The determination of these weight factors \( p_i \) depends in each case on an assessment, frequently only a very approximate one, of the quality of the observation. However, once the weight factors have been fixed, the formulae for the mean errors are

Mean error of unit weight:

\[
\mu = \sqrt{\frac{[vvp]}{n-1}};
\]

Mean error of the mean:

\[
\mu_L = \sqrt{\frac{[vvp]}{[p](n-1)}}.
\]

The mean is no longer equal to the arithmetic mean of the single measurements but is given by the expression

\[
L = \frac{1}{[p]} (l_1 p_1 + l_2 p_2 + \cdots + l_n p_n) = \frac{[lp]}{[p]}.
\]