9. SOME SCHEDULE ALGEBRA

9-1. Feasibility and Compatibility

In Section 7-3, we observed how the relationship between forward planning and backward planning in a class of scheduling problems was reflected by the relation of conjugacy \( A \leftrightarrow A^* \). With the notation of that section, suppose that the \((nxn)\) matrix \( A \) defines a process for which the vector of earliest permissible start-times for cycle nought is \( \mathbf{x} \in \mathbb{E}_n \). Then the vector of actual start-times for cycle nought will be some \( t(0) \geq \mathbf{x} \). So the vector of earliest permissible start-times for cycle one will be \( A \otimes t(0) \), and the vector of actual start-times for cycle one will be \( t(1) \geq A \otimes x \). By an obvious induction, the vector of actual start-times of cycle \( r \) will be \( t(r) \) where:

\[
\begin{align*}
\mathbf{x} &\geq A \otimes \mathbf{t}(r-1) \geq \cdots \geq A^r \otimes t(0) \geq A^r \otimes \mathbf{x} \\
\end{align*}
\] (9-1)

Suppose the process is due to run up to and including cycle \( N \), and must terminate at or before a given vector of finishing-times \( \mathbf{z} \). Evidently, this requires that \( A^{N+1} \otimes \mathbf{x} \leq \mathbf{z} \) and we shall call the pair \( x, z \) of vectors compatible for \( (A, N) \) if this holds. A sequence \( \{t(r)\} \) \((r=0, \ldots, N)\) will be called feasible for \( (x, z, N) \) if we have:

\[
\begin{align*}
(\text{i}) & \quad t(r+1) \geq A \otimes t(r) \\
(\text{ii}) & \quad t(N+1) \leq z \\
(\text{iii}) & \quad t(0) \geq x \\
\end{align*}
\] (9-2)

We shall require the following simple but important lemma.

Lemma 9-1. Let \( E \) be a pre-residuated belt which satisfies axiom \( X_{12} \). If \( A \in \mathcal{M}_{mn} \), \( x \in \mathbb{E}_n \) and \( z \in \mathbb{E}_m \), we have \( z \geq A \otimes x \) if and only if \( x \leq A^* \otimes z \).

Hence, if \( m=n \) and \( N>0 \), the following are all equivalent:

\[
\begin{align*}
\mathbf{z} &\geq A^{N+1} \otimes \mathbf{x}; A^* \otimes \mathbf{z} \geq A^N \otimes \mathbf{x}; \ldots; A^N \otimes \mathbf{z} \geq A \otimes \mathbf{x}; A^{(N+1)*} \otimes \mathbf{z} \geq \mathbf{z}. \\
\end{align*}
\]

Proof. If \( z \geq A \otimes x \) we have:

\[
\begin{align*}
A^* \otimes \mathbf{z} &\geq A^* \otimes (A \otimes \mathbf{x}) \\
&\geq x \quad \text{(by isotonicity)} \\
\end{align*}
\]

And dually, the converse holds.

Hence if \( z \geq A^{N+1} \otimes x \) with \( N \geq 0 \) we have:

\[
\mathbf{z} \geq A \otimes (A^N \otimes \mathbf{x})
\]

Hence \( A^* \otimes \mathbf{z} \geq A \otimes x \), and by iteration: \( A^r \otimes \mathbf{z} \geq A^{N+1-r} \otimes \mathbf{x} \), \((r=1, \ldots, N)\), and

\[
A^{(N+1)*} \otimes \mathbf{z} \geq \mathbf{z}
\]

And, dually, the converse sequence of implications holds.

We may now prove the following theorems in the theory of scheduling.

R. Cuninghame-Green, *Minimax Algebra*
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Theorem 9-2. Let E, receive the principal interpretation. Let $A \in \mathcal{N}_{\text{in}}$ and $\bar{x}, \bar{y} \in E_{\text{in}}$ for some integer $n \geq 1$. Let $N \geq 0$ be a given integer. Then the following conditions are equivalent:

(i) $x, y$ are compatible for $(A, N)$.

(ii) There exists a sequence $(t(r))$ which is feasible for $(x, y, N)$.

(iii) For all $r = 0, 1, \ldots, (N+1)$, there holds

$$\forall x \leq x^{(N-r+1)*} \leq x$$

(where $x^{(N-r+1)*} \leq x$ and $x^{(N-r+1)*} = x$ by definition when $r = 0, s = 0$).

Proof. Conditions (i) and (iii) are equivalent by Lemma 9-1. Also (9-2) clearly implies $x \geq x^{(N+1)} \geq x \oplus t(N) \geq \cdots \geq x^{(N+1)} \oplus t(0) \geq x^{(N+1)} \oplus x$, so condition (ii) implies condition (i). Finally, if $x \geq x^{(N+1)} \oplus x$ then the sequence $(t(r))$ defined by: $t(r) = x^{(N-r+1)*} \leq x$ and $x^{(N-r+1)*} = x$ by definition when $r = 0, s = 0$.\hfill \Box

Theorem 9-3. With the notation of Theorem 9-2, any sequence $t(r)$ which is feasible for $(x, y, N)$ satisfies:

$$\forall x \leq t(r) \leq x^{(N-r+1)*} \oplus x$$

Proof. From (i) of (9-2) we have for $0 \leq s \leq r \leq (N+1)$:

$$t(r) \geq x^{(r-s)} \oplus t(s)$$

So

$$t(r) \geq x^{(N+1)} \oplus x^{(0)}$$

And

$$t(N+1) \geq x^{(N-r+1)} \oplus t(r)$$

By Lemma 9-1, this last relation gives:

$$t(r) \leq x^{(N-r+1)*} \oplus t(N+1),$$

(9-4)

We may now use (ii) and (iii) of (9-2), in (9-4) and (9-3) respectively, to give (by isotonicity):

$$\forall x \leq x^{(N-r+1)*} \oplus x$$

\hfill \Box

9-2. The Float

Suppose at a certain moment that the first-following cycle which we must begin is cycle $s \leq 1 \leq s \leq N$, and that because of given circumstances the vector of earliest permissible start times for this cycle is $x$. The question arises: is the target vector of completion times $x$ still achievable? Formally, let $(t(r))$

$(r = 0, \ldots, (s-1))$ be a sequence satisfying:

(i) $t(o) \geq x$

(ii) If $s > 2$ then $t(r+1) \geq A \oplus t(r)$ $(r = 0, \ldots, (s-2))$

\hfill (9-5)