Part II. Continuous parameter

§ 1. Transition matrix: basic properties

In Part II it will be convenient to begin with the analysis of a denumerable set of real valued functions, later (in § 4) to be identified as the transition probability functions of a continuous parameter Markov chain. Although the questions treated in the next three sections are of a purely analytic nature some of the methods used will be probabilistically inspired. Occasionally the results are given in a more general form then is required by later applications.

Henceforth $T$ stands for the interval $[0, \infty)$, $T^0$ for $(0, \infty)$; $I$ for a denumerable set of indices, later to be specified to be the minimal state space of a Markov chain. Unless otherwise stated, the letters $s, t, u, \delta, \varepsilon$ denote positive real numbers, namely points of $T^0$; $i, j, k, l$ denote elements of $I$; $n, m, n$ denote positive integers. An unspecified sum over the indices is over all $I$.

A transition matrix is a finite or denumerable array of functions $(p_{ij}(\cdot))$ or more simply $(p_{ij}), i, j \in I$, defined on $T^0$ satisfying the following three conditions: for every $i, j$ and $s, t$,

(A) $p_{ij}(t) \geq 0$;
(B) $\sum_j p_{ij}(t) = 1$;
(C) $\sum_k p_{ik}(s) p_{kj}(t) = p_{ij}(s + t)$.

If we denote the matrix $(p_{ij}(t))$ by $P(t)$, then these properties can be stated as follows: each element of $P(t)$ is nonnegative; each row sum is equal to one; and the family $\{P(t), t \in T^0\}$ is a semigroup with respect to the usual matrix multiplication. (Convergence of the row-column product in (C) in the infinite case is ensured by (A) and (B).) Conditions (A) and (B) together may be expressed by saying that for each $t$, the matrix $(p_{ij}(t))$ is stochastic. Condition (C) is often referred to as the Chapman-Kolmogorov equation.

In the sequel measurability on $T$ or part of it means Lebesgue measurability, unless otherwise specified. This measure is denoted by $\mu$ and the phrases “almost all” and “almost everywhere”, abbreviated to “a.a.” and “a.e.”, have the usual meaning.
An example of a transition matrix whose elements are not measurable will be given at the end of this monograph. Measurability of all the elements of the matrix has far-reaching consequences. We record this as a basic assumption:

(D) Every $p_{ij}$ is a measurable function in $T^0$.

A transition matrix which satisfies the condition (D) is said to be measurable.

**Theorem 1.** For any transition matrix $(p_{ij})$ and each fixed $h>0$ the sum

$$\sum_j |p_{ij}(t + h) - p_{ij}(t)|$$

is nonincreasing as $t$ increases. If $(p_{ij})$ is measurable, then the above sum tends to zero uniformly in $t \geq \delta > 0$ as $h \to 0$. In particular, each $p_{ij}$ is uniformly continuous in $[\delta, \infty)$ for every $\delta > 0$.

**Proof.** We have if $0 < s < t$,

$$\sum_j |p_{ij}(t + h) - p_{ij}(t)| = \sum_j \sum_k |p_{ik}(s + h) - p_{ik}(s)| \cdot p_{kj}(t - s)$$

$$\leq \sum_k |p_{ik}(s + h) - p_{ik}(s)| \cdot \sum_j p_{kj}(t - s) = \sum_k |p_{ik}(s + h) - p_{ik}(s)| .$$

This proves the first assertion. Next if the matrix is measurable, then upon integrating we obtain, if $t \geq \delta > 0$,

$$\sum_j |p_{ij}(t + h) - p_{ij}(t)| \leq \sum_k \frac{1}{\delta} \int_0^\delta |p_{ik}(s + h) - p_{ik}(s)| \, ds .$$

Hence if $0 \leq h \leq \delta$ the first series is dominated by

$$\sum_k \frac{2}{\delta} \int_0^{2\delta} p_{ik}(s) \, ds$$

and consequently converges uniformly in $h \in [0, \delta]$ and $t \in [\delta, \infty)$. By a well known theorem (see e.g. TITCHMARSH [1, p. 371]) we have for each $k$,

$$\lim_{h \to 0} \int_0^\delta |p_{ik}(s + h) - p_{ik}(s)| \, ds = 0 .$$

The second assertion of the theorem follows from this and the uniform convergence, and the third assertion is an immediate consequence. \[\]

The most important part of Theorem 1 is: *every element $p_{ij}$ of a measurable transition matrix is a continuous function in $T^0$.*

In order to lead to a further essential hypothesis regarding the transition matrix, we study the behavior of $p_{ij}(t)$ as $t \to 0$ from the