we have, upon substituting into (10) and using the preceding equation,

\[ u_i \geq \sum_{k \neq j} p_{ik} \sum_{r=1}^{n} f_{ir}^{(r)} + f_{ij} = \sum_{r=1}^{n+1} f_{ij}^{(r)}. \]

Thus by induction the last inequality is true for all \( n \). Letting \( n \to \infty \) we obtain (11).

Notes. Theorem 1 (for a positive class of period one) seems to be due to Feller [3], although Kolmogorov [3] proved (2) and (3) without the intervention of (1). It should be noted, however, that it is feasible to state the theorem in the form given here, and not merely for a positive class. Otherwise the determination of a positive convergent solution \( \{u_i\} \) would not by itself prove that the class is positive. This would be more than a nuisance in practice, since the only general method of showing that a class is positive is precisely to solve the system (1).

Theorems 2 and 3 were given by the author in his lectures in 1950; see Loève [1]. Theorem 4 was proved by Kolmogorov [3] “geometrically” in the probabilistic form given after Theorem 4; he then deduced (2) and (3) from it. This approach has some independent interest.

For non-dissipative M.C.’s see Mauldon [1].

§ 8. Repetitive pattern and renewal process

This section is a digression. Its purpose is to establish the equivalence of several notions in current usage.

Let \( \{\Omega, \mathcal{F}, P\} \) be a probability triple and let \( \{z_n, n \geq 1\} \) be a sequence of discrete random variables with state space \( I \). Suppose that there is a single-valued function \( R \) defined for every finite sequence of states and assuming only the values 0 and 1. We say that the \( z_n \) process possesses the repetitive pattern determined by \( R \), iff the following conditions are satisfied:

(a) if \( R(i_1, \ldots, i_m) = 0 \) then \( R(i_1, \ldots, i_m, i_{m+1}, \ldots, i_{m+n}) = 0 \) if and only if \( R(i_{m+1}, \ldots, i_{m+n}) = 0 \);

(b) if \( R(i_1, \ldots, i_m) = 0 \) and \( R(i_1, \ldots, i_m, i_{m+1}, \ldots, i_{m+n}) = 0 \), then

\[
P\{z_v(\omega) = i_v, \ 1 \leq v \leq n + m\} = P\{z_v(\omega) = i_v, \ 1 \leq v \leq m\} P\{z_v(\omega) = i_{m+v}, \ 1 \leq v \leq n\}.
\]

The repetitive pattern is said to occur to the process at time \( n \) iff

\[ R(z_1(\omega), \ldots, z_n(\omega)) = 0. \]
It is natural to associate with the \( \{ z_n \} \) process another process \( \{ y_n, n \geq 0 \} \) defined as follows:

\[
y_0(\omega) = 0, \quad y_n(\omega) = R\left( z_1(\omega), \ldots, z_n(\omega) \right), \quad n \geq 1.
\]

Thus the repetitive pattern occurs to the \( \{ z_n \} \) process at time \( n \) if and only if \( y_n(\omega) = 0 \). It follows from (a) and (b) that we have

\[
\begin{align*}
P \{ y_{m+n}(\omega) = 1, \ 1 \leq v < n; \ y_n(\omega) = \varepsilon, \ 0 \leq v < m; \ y_m(\omega) = 0 \} &= P \{ y_v(\omega) = 1, \ 1 \leq v < n; \ y_n(\omega) = 0 \}, \\
P \{ y_v(\omega) = 1, \ 1 \leq v < n; \ y_n(\omega) = 0 \} &= P \{ y_v(\omega) = 1, \ 1 \leq v < m; \ y_m(\omega) = 0 \} \\
\end{align*}
\]

for every \( m \geq 0, n \geq 1 \) and every set of \( \varepsilon \) for which the conditional probability is defined. If the \( \{ y_n \} \) process is a M.C. then the condition (1) is clearly satisfied. It is easy to see that in general (1) is less stringent than the Markov property. We call a sequence of random variables \( \{ y_n, n \geq 0 \} \) with \( y_0 = 0 \), assuming only the values 0 and 1 and satisfying the condition (1), a renewal process. Thus a repetitive pattern generates a renewal process. On the other hand, the renewal process \( \{ y_n \} \) possesses a natural repetitive pattern determined by \( R(\varepsilon_1, \ldots, \varepsilon_m) = \varepsilon_m \). We note that it follows from (1) that

\[
\begin{align*}
P \{ y_{m+n}(\omega) = 1, \ 1 \leq v \leq n; \ y_v(\omega) = \varepsilon, \ 0 \leq v < m; \ y_m(\omega) = 0 \} &= P \{ y_v(\omega) = 1, \ 1 \leq v \leq n \}, \\
P \{ y_v(\omega) = 1, \ 1 \leq v \leq n \} &= P \{ y_v(\omega) = 1, \ 1 \leq v < n; \ y_n(\omega) = 0 \}. \\
\end{align*}
\]

Let

\[
\begin{align*}
p_n &= P \{ y_n(\omega) = 0 \}, \\
f_n &= P \{ y_v(\omega) = 1, \ 1 \leq v < n; \ y_n(\omega) = 0 \}, \quad n \geq 1. \\
f_\infty &= P \{ y_v(\omega) = 1, \ 1 \leq v < \infty \}. \\
\end{align*}
\]

It follows from (1) by an induction on \( n \) that

\[
p_n = P \{ y_{m+n}(\omega) = 0 \} | y_v(\omega) = \varepsilon, \ 0 \leq v < m; \ y_m(\omega) = 0 \}.
\]

We can now prove, exactly as formula (6.1), the renewal equation:

\[
\begin{align*}
p_0 &= 1, \\
p_n &= \sum_{v=1}^{n} f_v p_{n-v}, \quad n \geq 1.
\end{align*}
\]

It is thus clear that the analogues of Theorems 5.4 and 6.1 must also hold since their proofs depend only on (6.1). This is a purely algebraic-analytical point of view concerning the numbers \( \{ p_n \} \) and \( \{ f_n \} \). However, we can go further and associate a M.C. with the renewal process in such a way that a renewal becomes identifiable with the entrance into a certain fixed state.