6.1 Introduction

Domain discretization is one of the most important steps in many numerical methods to solve boundary value problems. In finite element method (FEM), the whole domain is discretized by elements. In boundary element method (BEM) the boundary of the domain is discretized by elements. The choice of the geometry of the element and the form of the approximating function to represent the behaviour of field variables (such as the potential \( \phi \), field strength \( E \text{ or } H \) and so on) within each element are both extremely important. They strongly influence the accuracy of the results, the computing time and the software engineering of computer programs. Hence the problem of element discretization is a generic problem and is common to element approximate methods as well as FEM. Hence element discretization techniques are discussed in one chapter.

It has been shown in former chapters (4 and 5) that a matrix equation is used for the solution of partial differential equations. The coefficients of the matrix are calculated via the partial derivatives of the shape functions. Hence the expressions of shape functions of various elements are the first to be formulated in the use of FEM. The shape functions of a 3-node triangular element, corresponding to linear approximation, has been introduced in Chaps. 4 and 5 to convey the principles underlying FEM. To obtain more accurate results and to solve different problems, many different kinds of elements and approximating functions can be used. For instance, both theory and experience indicate that, for many two dimensional problems, it is best to subdivide the problem region into the smallest possible number of large triangles, and to achieve the desired accurate solution by using high-order polynomial approximations on this very coarse mesh.

This chapter intends to provide the principles required to construct the various shape functions belonging to Lagrange and Hermite polynomials. For easy to evaluate element coefficients, the shape functions of different elements are formulated in local coordinates. The most effective elements belonging to the isoparametric family with linear and high order interpolations are the main topic in this chapter.
6.2 Types and requirements of the approximating functions

The functions used to represent the behaviour of field variables specified by the governing equations are called interpolation functions or approximating functions. In element discretization methods, the approximating function $u(x, y)$ is defined for each element. It is expressed by the following equation:

$$u(x, y) \approx \tilde{u}(x, y) = \sum_{k=1}^{m} \alpha_k \psi_k(x, y) = \sum_{k=1}^{m} \alpha_k N_k(x, y)$$  \hspace{1cm} (6.2.1)

In Eq. (6.2.1), $\alpha_k$ are undetermined parameters. They may be the value of potentials or other physical quantities defined in a governing equation, hence they are called generalized coordinates. As shown in Chaps. 4 and 5, $\alpha_k$ are nodal values of a potential. $\psi_k$ is a set of independent basis functions. Thus, the approximating function is a linear combination of a set of discretized values and the basis functions.

In the method using weighted residuals, the interpolation function needs to ensure that

$$\int_{\Omega} RW \, d\Omega = 0$$ \hspace{1cm} (6.2.2)

where $R$ is the residual due to the approximation and $W$ is a weighting function. This equation guarantees that the integral of the weighted residuals over the whole domain approaches zero.

In the variational principle, the generalized coordinates are determined by the extremum principle:

$$\frac{\partial I(u)}{\partial \alpha_k} = 0$$ \hspace{1cm} (6.2.3)

where $I(u)$ is the equivalent functional of the problem to be solved. In other words, these two methods have to ensure that the approximate solution must be convergent to the exact solution. Hence the interpolation function needs to satisfy some requirements discussed in the following subsection.

6.2.1 Lagrange and Hermite shape functions

In FEM, there are two categories of elements known as Lagrange and Hermite. The former takes the values of the field variables at nodes of the element as being the unknowns. The latter takes both the unknown function and its partial derivative as unknowns. Thus, a Hermite element contains more than one unknown at each node. Usually the symbol $N_f$ is used to represent the degrees of freedom (in structural machanics, the number of unknowns is called the degrees of freedom) at each node. $N_f = 1$ represents the Lagrange element, $N_f > 1$ indicates the