25. Supermartingales and Submartingales

In Chapter 24 we defined a martingale via an equality for certain conditional expectations. If we replace that equality with an inequality we obtain supermartingales and submartingales. Once again $(\Omega, \mathcal{F}, P)$ is a probability space that is assumed given and fixed, and $(\mathcal{F}_n)_{n \geq 1}$ is an increasing sequence of $\sigma$-algebras.

**Definition 25.1.** A sequence of random variables $(X_n)_{n \geq 0}$ is called a submartingale (respectively a supermartingale) if

(i) $E\{|X_n|\} < \infty$, each $n$;

(ii) $X_n$ is $\mathcal{F}_n$-measurable, each $n$;

(iii) $E\{X_n | \mathcal{F}_m\} \geq X_m$ a.s. (resp. $\leq X_m$ a.s.) each $m \leq n$.

The sequence $(X_n)_{n \geq 0}$ is a martingale if and only if it is a submartingale and a supermartingale.

**Theorem 25.1.** If $(M_n)_{n \geq 0}$ is a martingale, and if $\varphi$ is convex and $\varphi(M_n)$ is integrable for each $n$, then $(\varphi(M_n))_{n \geq 0}$ is a submartingale.

**Proof.** Let $m \leq n$. Then $E\{M_n | \mathcal{F}_m\} = M_m$ a.s., so $\varphi(E\{M_n | \mathcal{F}_m\}) = \varphi(M_m)$ a.s., and since $\varphi$ is convex by Jensen's inequality (Theorem 23.9) we have

$$E\{\varphi(M_n) | \mathcal{F}_m\} \geq \varphi(E\{M_n | \mathcal{F}_m\}) = \varphi(M_m).$$

**Corollary 25.1.** If $(M_n)_{n \geq 0}$ is a martingale then $X_n = |M_n|$, $n \geq 0$, is a submartingale.

**Proof.** $\varphi(x) = |x|$ is a convex, so apply Theorem 25.1.

**Theorem 25.2.** Let $T$ be a stopping time bounded by $C \in \mathbb{N}$ and let $(X_n)_{n \geq 0}$ be a submartingale. Then $E\{X_T\} \leq E\{X_C\}$.

**Proof.** The proof is analogous to the proof of Theorem 24.2, so we omit it.

The next theorem shows a connection between submartingales and martingales.
Theorem 25.3 (Doob Decomposition). Let \( X = (X_n)_{n \geq 0} \) be a submartingale.

There exists a martingale \( M = (M_n)_{n \geq 0} \) and a process \( A = (A_n)_{n \geq 0} \) with \( A_{n+1} \geq A_n \) a.s. and \( A_{n+1} \) being \( F_n \)-measurable, each \( n \geq 0 \), such that

\[
X_n = X_0 + M_n + A_n, \quad \text{with } M_0 = A_0 = 0.
\]

Moreover such a decomposition is a.s. unique.

Proof. Define \( A_0 = 0 \) and

\[
A_n = \sum_{k=1}^{n} E\{X_k - X_{k-1} | F_{k-1}\} \quad \text{for } n \geq 1.
\]

Since \( X \) is a submartingale we have \( E\{X_k - X_{k-1} | F_{k-1}\} \geq 0 \) each \( k \), hence \( A_{k+1} \geq A_k \) a.s., and also \( A_{k+1} \) being \( F_k \)-measurable. Note also that

\[
E\{X_n | F_{n-1}\} - X_{n-1} = E\{X_n - X_{n-1} | F_{n-1}\} = A_n - A_{n-1},
\]

and hence

\[
E\{X_n | F_{n-1}\} - A_n = X_{n-1} - A_{n-1};
\]

but \( A_n \in F_{n-1} \), so

\[
E\{X_n - A_n | F_{n-1}\} = X_{n-1} - A_{n-1}. \tag{25.1}
\]

Letting \( M_n = X_n - A_n \) we have from (25.1) that \( M \) is a martingale and we have the existence of the decomposition.

As for uniqueness, suppose

\[
X_n = X_0 + M_n + A_n, \quad n \geq 0,
\]

\[
X_n = X_0 + L_n + C_n, \quad n \geq 0,
\]

are two such decompositions. Subtracting one from the other gives

\[
L_n - M_n = A_n - C_n. \tag{25.2}
\]

Since \( A_n, C_n \) are \( F_{n-1} \) measurable, \( L_n - M_n \) is \( F_{n-1} \) measurable as well; therefore

\[
L_n - M_n = E\{L_n - M_n | F_{n-1}\} = L_{n-1} - M_{n-1} = A_{n-1} - C_{n-1} \quad \text{a.s.}
\]

Continuing inductively we see that \( L_n - M_n = L_0 - M_0 = 0 \) a.s. since \( L_0 = M_0 = 0 \). We conclude that \( L_n = M_n \) a.s., whence \( A_n = C_n \) a.s. and we have uniqueness.

Corollary 25.2. Let \( X = (X_n)_{n \geq 0} \) be a supermartingale. There exists a unique decomposition

\[
X_n = X_0 + M_n - A_n, \quad n \geq 0
\]

with \( M_0 = A_0 = 0, (M_n)_{n \geq 0} \) a martingale, and \( A_k \) being \( F_{k-1} \)-measurable with \( A_k \geq A_{k-1} \) a.s.