Chapter VIII. Atkin–Lehner Theory

Atkin-Lehner [A-L] showed how to construct in a natural way a basis for the space of modular forms of given level which are eigenfunctions for the Hecke operators prime to that level, satisfying the same formalism as for level 1. They worked on $\Gamma_0(N)$. Miyake [Mi] extended this to the general case, including the modular forms in the sense of Langlands in the context of representation theory. See also Casselman [Ca]. More recently, Li [Li] reconsidered the matter in the style of Atkin-Lehner, following [A–L] very closely.

Since the study of $\Gamma(N)$ can be reduced to that of $\Gamma_1(N)$ by conjugation, we give an exposition of the Atkin-Lehner theory on $\Gamma_1(N)$.

§ 1. Changing Levels

Let $d$ be a positive integer, $d > 1$, and $d | N$. We want to define two maps from $\mathcal{F}_1(N/d, k)$ to $\mathcal{F}_1(N, k)$. The sum of the images of these maps will be called the non-primitive subspace ("old" subspace) of $\mathcal{F}_1(N, k)$. As usual, the maps are defined first on the modular set.

We define $\pi_1(d): \mathcal{L}_1(N) \rightarrow \mathcal{L}_1(N/d)$ by

$$\pi_1(d):(t, L) \mapsto (dt, L).$$

Then on modular forms we have

$$\pi_1(d)k F(t, L) = F(dt, L).$$

Since $d/N = 1/(N/d)$, it follows immediately from the definitions that on the corresponding function $f(\tau)$, the operator $\pi_1(d)_k$ acts like the identity mapping. In other words, any function on the upper half plane invariant under $\Gamma_1(N/d)$ is also invariant under $\Gamma_1(N)$, and thus our operator $\pi_1(d)_k$ corresponds to this natural injection. In particular, on the $q$-expansion, the map is also the identity. We may write this in the form

$$\pi_1(d)_\infty = \text{id}.$$

We have a second map

$$\pi_2(d): \mathcal{L}_1(N) \rightarrow \mathcal{L}_1(N/d)$$

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given by
\[(t, L) \mapsto (t, L_{(N/d)\mathcal{R}}),\]
where \(L_{(N/d)\mathcal{R}}\) is the lattice generated by \(L\) and \((N/d)t\). Thus \(t\) has period \(N/d\) with respect to \(L_{(N/d)\mathcal{R}}\). Then the corresponding map on modular forms is determined by the computation:

\[
\pi_2(d)_k F\left(\frac{1}{N}, \tau, 1\right) = F\left(\frac{1}{N}, \left[\tau, \frac{1}{d}\right]\right)
= F\left(\frac{1}{N}, \frac{1}{d} [d\tau, 1]\right)
= d^k F\left(\frac{1}{Nd}, [d\tau, 1]\right).
\]

Thus we find:

**Lemma 1.** Let \(f\) be the function of \(\tau\) associated with \(F\). Then

\[
\pi_2(d)_k f(\tau) = d^kf(d\tau),
\]
and thus on the \(q\)-expansions,

\[
\pi_2(d)_\infty = d^k V_d,
\]
or in other words,

\[
(\pi_2(d)_k F)_\infty(q) = d^k F_\infty(q^d).
\]

The operator \(V_d\) had been defined in Chapter VII, § 3.

**Lemma 2.** (i) Let \(d_1, d_2 > 1\) be such that \(d_1d_2 \mid N\). Then:

\[
\pi_i(d_1d_2) = \pi_i(d_1)\pi_i(d_2) \quad \text{for } i = 1, 2.
\]

(ii) If \(d, d' > 1\) and \(dd'\) divides \(N\), then \(\pi_1(d)\) commutes with \(\pi_2(d')\).

**Proof.** Clear.

The lemma also applies to the modular forms by composition.

From Lemma 1, we see that the operators \(\pi_1(d)_k\) and \(\pi_2(d)_k\) map cusp forms into cusp forms. We let

\[
S_i^+(N, k)
\]