Chapter 1
Portfolio Optimization: An Overview

Abstract. In this chapter, we present a brief overview of portfolio optimization. First, we discuss the classical mean-variance model of portfolio optimization developed by Markowitz. We then discuss the various extensions of the Markowitz’s model by considering alternative measures of risk, namely, semivariance, absolute deviation and semi-absolute deviation.

1.1 Mean-Variance Model

‘Do not put all your eggs in one basket’ is an age old wisdom capturing the fundamental idea underlying portfolio optimization. The ‘wisdom’ essentially lies in the return-risk characteristics of the various assets. Clearly, ‘more’ assets may not necessarily be ‘good’ if all the assets exhibit the same return-risk characteristics. A ‘good’ portfolio is the one that gives higher return for a given level of risk or the one that gives lower risk for a given level of return. Thus, a ‘good’ portfolio would comprise assets that are different rather than similar in terms of these characteristics. Operationalization of the age old wisdom necessitated mathematical modeling for portfolio optimization. The mathematical problem of portfolio optimization can be formulated in many ways but the principal problems can be summarized as follows:

(i) Minimize risk for a specified expected return
(ii) Maximize the expected return for a specified risk
(iii) Minimize the risk and maximize the expected return using a specified risk aversion factor
(iv) Minimize the risk regardless of the expected return
(v) Maximize the expected return regardless of the risk

The solutions of the first three problems are called mean-variance efficient solutions. The fourth problem gives minimum variance solutions which are desirable for conservative investors. It is also used for comparison and benchmarking of other portfolios. The fifth problem gives the upper bound of the expected return which can be attained; this is also useful for comparisons.
Harry Markowitz made the major breakthrough in 1952 with the publication on portfolio selection theory [90]. The Markowitz’s theory, popularly referred to as modern portfolio theory, provided an answer to the fundamental question: How should an investor allocate capital among the possible investment choices? Markowitz suggested that it is impossible to derive all possible conclusions concerning portfolios. A portfolio analysis must be based on some criteria which serve as a guide to the important and unimportant, the relevant and irrelevant. The proper choice of criteria depends on the nature of the investor. For each type of investor the details of the portfolio analysis must be suitably selected. However, the two criteria that are common to all investors are expected (mean) return and variance of return (risk). Markowitz assumed that ‘beliefs’ or projections about assets follow the same probability rules that random variables obey. From this assumption, it follows that (i) the expected return on the portfolio is a weighted average of the expected returns on individual assets, and (ii) the variance of return on the portfolio is a particular function of the variances of and the covariances between assets, and their weights in the portfolio. Hence, investors must consider risk and return together and determine the allocation of capital among investment alternatives on the basis of the trade-off between them.

Further, Markowitz suggested that portfolio selection should be based on reasonable beliefs about future rather than past performances per se. Choices based on past performances alone assume, in effect, that average returns of the past are good estimates of the ‘likely’ return in the future; and variability of return in the past is a good measure of the uncertainty of return in the future.

In what follows next, we present the mathematical formulation of the mean-variance model proposed by Markowitz [90]. Let $R_i$ be a random variable representing the rate of return (per period) of the $i$-th asset $(i = 1, 2, \ldots, n)$. Also, let $x_i$ be the proportion of the total funds invested in the $i$-th asset.

**Definition 1.1 (Asset return).** The asset return is expressed as the rate of return which is defined during a given period as

$$
\left( \frac{((\text{closing price for the current period}) - (\text{closing price for the previous period}) + (\text{dividend(s) for the current period}))}{(\text{closing price for the previous period})} \right)
$$

Note that the period of return may be a day or a week or a month or a year.

In particular, for the $i$-th asset the realization $r_{it}$ of the random variable $R_i$ during period $t$ ($t = 1, 2, \ldots, T$) is defined as

$$
r_{it} = \frac{(p_{it}) - (p_{it-1}) + (d_{it})}{(p_{it-1})},
$$