Heat Conduction as Eigenvalue Problem

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Abstract. The time dependent solution of the heat conduction equation is represented by expanding $T$ in terms of eigenfunctions and eigenvalues ($=$ inverse time constants). Once these are known, the time evolution of $T$ can be computed readily for arbitrary heat sources. A method is presented for computing eigenfunctions and eigenvalues for a multilayer structure, where all layers have the same lateral extension and differ vertically in thickness and material parameters. Those structures are encountered in modeling IC packages and multichip modules in power electronic applications. Examples will be given for the temperature evolution calculated for the time dependent heat distribution in power converter applications. However, computation time and storage demands will be high if high spatial and temporal resolution for the temperature field is needed. Using effective time constants and functions the numerical effort can be reduced considerably.

1 Introduction

The thermal evolution in electronic systems for arbitrary heat generation may be computed by finite element (FEM) \cite{1}, finite differences or finite volume (FVM) methods. However, for models with a large number of grid points (83,100 grid points for the set-up of Fig. 1), CPU-time and storage requirements become to excessive for the transient calculation in case of complicated power pulses and load cycles over large time intervals.

In recent work \cite{3,4}, the time-dependent solution $T(x,t)$ of the heat conduction equation

\begin{equation}
\rho(x)c(x)\frac{\partial T(x,t)}{\partial t} = H(x,t) + \nabla_x \cdot (\lambda(x) \nabla_x T(x,t))
\end{equation}

was represented by expanding $T$ in terms of eigenfunctions and eigenvalues of (1) similar to the treatment of the quantum mechanical one particle Schroedinger equation. This is possible, provided that the mass density $\rho$, the specific heat $c$ and the thermal conductivity $\lambda$ of the structure depend only on the position $x$ and not on the temperature $T$ itself, i.e. (1) is linear in $T$. The letter $x$ denotes the 3D location of the space point $x,y,z$.

Once the eigenfunctions and eigenvalues of (1) are known, which do not depend on the heat generation density $H(x,t)$ but only on the geometry of the set-up, the boundary conditions, and the material parameters, the temporal evolution of $T$ can be computed very fast for arbitrary heat sources.
However, it is a non-trivial task to determine the eigenfunctions and eigenvalues for a complex three-dimensional structure with given boundary conditions. The problem could be solved by FEM-computation, similar to a mechanical modal analysis for determining the vibration characteristics (natural frequencies and mode shapes) of machine components. However, since the superposition of several thousand eigenfunctions is needed for the precise representation of the temperature field, again large computation times would result. This paper will review an algorithm for computing eigenfunctions and eigenvalues for a rectangular multilayer structure, where all layers have the same lateral extension and differ vertically by thickness and material parameters.

2 Theoretical Background

In order to calculate the eigenfunctions and eigenvalues for (1) with zero heat generation $H$, a separation of variables is introduced. When $T(x, t) = \psi(x) \cdot \vartheta(t)$ equation (1) with heat $H = 0$ can be rewritten as:

$$\frac{1}{\vartheta(t)} \frac{d\vartheta(t)}{dt} = \frac{1}{\rho(x)c(x)\psi(x)} \nabla_x \cdot (\lambda(x)\nabla_x \psi(x)) .$$

Since both sides of the equation depend on different variables they must be constant, where the constant $v = 1/\tau$ is an eigenvalue of the problem. Therefore: $\vartheta(t) = \vartheta(0)\exp(-t/\tau)$ and

$$\frac{1}{\tau} \psi(x) = \frac{-1}{\rho(x)c(x)} \nabla_x \cdot (\lambda(x)\nabla_x \psi(x)) .$$

The eigenvalue spectrum $v_i = 1/\tau_i$ of the last equation depends on the boundary conditions the eigenfunctions $\psi_i(x)$ are subjected to. As detailed in [3] the problem of determining the eigenvalue spectrum is similar to that of the quantum mechanical Schroedinger equation for one particle of mass $m$ moving in a rectangular potential wall. In the same way as for this problem the eigenvalues for (1) with $H = 0$ are real, positive, discrete (lumped) and not bounded from above.