A Software Implementation of Niederreiter–Xing Sequences

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Abstract. In a series of papers, Niederreiter and Xing introduced new construction methods for low-discrepancy sequences, more specifically (t, s)-sequences. As these involve the rather abstract theory of algebraic function fields — a special case of algebraic geometry and also closely related to function theory and algebraic number theory — for a long time no computer implementation of this new method was given. In this paper we present our efforts in this direction, address the algorithmical problems and give some numerical data obtained from our implementation.

1 Introduction

It is known that the minimal order of discrepancy for the first N points of an s-dimensional sequence is at most $\mathcal{O}((\log N)^s/N)$ as N increases. Sequences that attain this bound are called low-discrepancy sequences. An especially fruitful construction method using digit expansions leads to so-called digital (t, s)-sequences constructed over $\mathbb{F}_b$, where b is some prime power. The Sobol', Faure and Niederreiter sequences are examples of increasing generality. Well-known low-discrepancy sequences that are not (t, s)-sequences are the good lattice-points and the Halton sequence.

All of these have a discrepancy upper bound of $\mathcal{O}((\log N)^s/N)$ (we refer to star discrepancy throughout this section), but differ in the implied constant of the upper bound, whose asymptotic orders with respect to s are shown in Table 1. The quantity t is an integer parameter that describes the distribution

Sequence	Discrepancy bound constant
Good Lattice Points	2^s
Halton sequence	$s!$
(t, s)-sequence constr. over $\mathbb{F}_b$	$\frac{b^s b^t}{s!(2 \log b)^s} \approx b^{t(s) - s \log_b s}$

Table 1. Comparison of constants

quality of a given (t, s)-sequence.

It should be noted that in a preprint, Atanassov [1] showed that the constant for the Halton sequence is of the smaller order $\mathcal{O}((2^s \log s)^{-1})$.

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An important application of low-discrepancy sequences is in the quasi-Monte Carlo method of very high-dimensional numerical integration. Therefore the asymptotic behaviour of the discrepancy upper bound constants with respect to the dimension s is of interest. The behavior of (t, s)-sequences depends on the quality parameter t, which is an increasing function in s. For the best previously known construction method, the Niederreiter sequences, $t(s)$ is of the order $\mathcal{O}(s \log_b s)$. This has been dramatically improved by Niederreiter and Xing in a series of papers, where they obtain $t(s) \in \mathcal{O}(s)$ by use of algebraic geometry. The ensuing constant hence is of order $\mathcal{O}((b/s)^s)$, which even improves Atanassov's bound for the Halton sequence. For this reason Niederreiter-Xing (NX-)sequences can be considered as the currently optimal low-discrepancy sequences.

Note that not only the low constant implies the practical relevance of NX-sequences, but also the fact that there is a fixed base b for increasing dimension s, which is not the case if we employ Faure sequences. For fixed b, there are better bounds on the integration error of certain function classes (Korobov classes with respect to Walsh functions) and also computer implementation can benefit from a fixed base, especially if b equals a power of 2.

In the next section we are going to introduce some notions of function field theory that we need to describe the algorithmic issues of a computer implementation in Section 3. Following that, some numerical results are given in Section 4, and a brief outlook to further developments in Section 5.

2 Definitions

2.1 Function field theory

In the following paragraphs we are going to describe some concepts of algebraic function field theory in a very brief and not altogether rigorous manner. The strict definitions and proofs of the statements can be looked up, e.g. in [2,3] (see also [13]).

An *algebraic function field* F/K is a finite algebraic extension $K(x, y)$ of the field of rational functions $K(x)$. Here we will always assume that K is some finite field $\mathbb{F}_b$.

In a rational function field (the field of rational functions), a valuation can be defined for each irreducible polynomial (recall that an integer function $v(\cdot)$ is called a *valuation* iff $c^{v(\cdot)}$ is a norm for an arbitrary $c \in (0, 1)$): for a rational function $f \in K(x)$ and an irreducible polynomial $p \in K[x]$, the integer $v_p(f)$ shall be defined as the exponent of p in the unique factorization of f into integer powers of irreducible polynomials (as special case, $v_p(0) := \infty$). Then the function v_p is a valuation of $K(x)$. Apart from the v_p, only one further valuation exists, namely $v_\infty(f) := \deg(f_{\mathrm{den}}) - \deg(f_{\mathrm{num}})$, where $f_{\mathrm{num}}/f_{\mathrm{den}}$ is the unique representation of f as a fraction of coprime polynomials.