12. Removal of Resonances

From the perturbative procedure in the last chapter we have learned that in the proximity of resonances of the unperturbed system, resonant denominators appear in the expression for the adiabatic invariants. We now wish to begin to locally remove such resonances by trying, with the help of a canonical transformation, to go to a coordinate system which rotates with the resonant frequency.

Let the unperturbed, solved problem with two degrees of freedom be given by

\[ H_0 = \frac{1}{2} \sum_{k=1}^{2} \left( p_k^2 + \omega_k^2 q_k^2 \right) . \]  

(12.1)

The transition to action-angle variables \( J_i, \theta_i \) is achieved with the transformation

\[ q_i = \sqrt{\frac{2J_i}{\omega_i}} \cos \theta_i , \]

(12.2)

\[ p_i = -\sqrt{2\omega_i J_i} \sin \theta_i . \]

(12.3)

These formulae agree with (10.28/29) in so far as we have replaced \( \theta \) by \( \theta + \pi/2 \) there. This corresponds to a simple phase change in \( \theta_i = \omega_i t + \beta_i \). Furthermore, it holds that \( J_i = (1/2\pi) \int p_i \, dq_i \). Thus we can write (12.1) as

\[ H_0(J_i) = \omega_1 J_1 + \omega_2 J_2 , \quad \omega_i = \frac{\partial H_0}{\partial J_i} . \]  

(12.4)

Let the perturbation term be given by

\[ H_1 = q_1^2 q_2 - \frac{1}{3} q_2^3 . \]  

(12.5)

and let us assume a 1:2 resonance between \( \omega_1 \) and \( \omega_2 \), i.e., that oscillator 1 is slower than oscillator 2. Then our complete Hamiltonian reads

\[ H = \frac{1}{2} \left( p_1^2 + p_2^2 \right) + \frac{1}{2} q_1^2 + 2q_2^2 + q_1^2 q_2 - \frac{1}{3} q_2^3 \]  

(12.6)

with

\[ \omega_1 = 1 , \quad \omega_2 = 2 . \]
The resonance of the unperturbed frequencies,

$$\frac{\omega_2}{\omega_1} = \frac{r}{s} = \frac{2}{1} \tag{12.7}$$

leads to divergent expressions in the perturbative solution of the problem. We shall therefore attempt to eliminate the commensurability (12.7),

$$r\omega_1 - s\omega_2 = 0 \tag{12.8}$$

by making a canonical transformation to new action-angle variables $\hat{J}_1, \hat{\theta}_1$, so that only one of the two actions $\hat{J}_i$ appears in the new, unperturbed Hamiltonian. In order to do so, we choose the generating function

$$F_2 = (r\hat{\theta}_1 - s\hat{\theta}_2)\hat{J}_1 + \theta_2\hat{J}_2 \tag{12.9}$$

The corresponding transformation equations then read

\[
\begin{align*}
J_1 &= \frac{\partial F_2}{\partial \hat{\theta}_1} = r\hat{J}_1 = 2\hat{J}_1 \\
J_2 &= \frac{\partial F_2}{\partial \hat{\theta}_2} = \hat{J}_2 - s\hat{J}_1 = \hat{J}_2 - \hat{J}_1 \\
\hat{\theta}_1 &= \frac{\partial F_2}{\partial J_1} = r\theta_1 - s\theta_2 = 2\theta_1 - \theta_2 \\
\hat{\theta}_2 &= \frac{\partial F_2}{\partial J_2} = \theta_2
\end{align*}
\tag{12.10}
\]

This choice of coordinates puts the observer into a coordinate system in which the change of $\hat{\theta}_1$,

$$\dot{\hat{\theta}}_1 = r\hat{\theta}_1 - s\hat{\theta}_2 = r\omega_1 - s\omega_2 \tag{12.12}$$

measures small deviations from the resonance (12.8). For $\hat{\theta}_1 = 0$, the system is in resonance. The variable $\hat{\theta}_1$ changes slowly and is, in the resonant case, a constant. Thus $\hat{\theta}_2$ is the fast variable, and we shall average over it.

One should note that the new Hamiltonian is now actually only dependent on a single action variable, i.e., $\hat{J}_2; \hat{J}_1$ does not appear:

$$H_0 = (10) \omega_1(2\hat{J}_1) + \omega_2(\hat{J}_2 - \hat{J}_1) = \frac{\omega_2}{2}(2\hat{J}_1) + \omega_2(\hat{J}_2 - \hat{J}_1) = \omega_2\hat{J}_2 \tag{12.13}$$

The perturbation term is then

\[
\begin{align*}
\varepsilon H_1 &= \varepsilon q_1^2 q_2 - \frac{\varepsilon}{3} q_2^3 \\
&= \varepsilon \left( \frac{2J_1}{\omega_1} \right) \left( \frac{2J_2}{\omega_2} \right)^{1/2} \cos^2 \theta_1 \cos \theta_2 - \frac{\varepsilon}{3} \left( \frac{2J_2}{\omega_2} \right)^{3/2} \cos^3 \theta_2
\end{align*}
\]