XI. The Serre Spectral Sequence

In this chapter we study the Serre spectral sequence \( \{ E_r^{s,t}(p), d_r \} \) over \( K = \mathbb{Z} \), and \( \mathbb{Z}_2 \), of the path space fibration

\[ p : \mathcal{P}(M) \to M, \]

with \( M = \mathbb{F}_k(\mathbb{R}^{n+1}) \) or \( \mathbb{F}_{k+1}(S^{n+1}) \). Here the paths are based at an appropriate basepoint.

First, in §1 we take up the case of \( M = \mathbb{R}^{n+1} \). We shall see that the spectral sequence stabilizes at the \( n \)th term, in the sense that

\[ E_r^{n+1}(p) \cong E_\infty^{n+1}(p) \cong K. \]

Consequently, regarding \( H_\ast(\Omega(M); K) \) as a chain algebra, with the trivial differential and \( K \) as a trivial chain module over it, we interpret the \( E_\ast^{n+1} \) term of the spectral sequence as an acyclic, free resolution of \( K \) over \( H_\ast(\Omega(M); K) \). This result, together with the fact that \( H_\ast(\Lambda(M); \mathbb{Z}) \cong \text{Tor}^{H_\ast(\Omega(M); \mathbb{Z})}(H_\ast(\Omega(M); \mathbb{Z}); K) \) (see §6 of Chapter IX), is the primary tool used in Chapter XII to study the module \( H_\ast(\Lambda(M); \mathbb{Z}_2) \).

In §§2 and 3 we take up the case when \( M = S^{n+1} \). The situation here is somewhat different, and the results of §1 are adapted suitably.

1 The Case of \( \mathbb{F}_{k-r,r}, n > 1 \)

In keeping with the notation of Chapter II, put

\[ \mathbb{F}_{r,k-r} = \mathbb{F}_r(\mathbb{R}^{n+1} - Q_{k-r}), \quad \mathbb{R}_{k-r}^{n+1} = \mathbb{R}^{n+1} - Q_{k-r}, \]

and consider the path fibration

\[ p_{k-r,r} : \mathcal{P}(\mathbb{F}_{k-r,r}) \to \mathbb{F}_{k-r,r} \]

that sends a based Moore path \((\alpha, r)\) to its endpoint \( \alpha(r) \). Throughout this section, we assume that the homology and cohomology groups are with integral coefficients.
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Theorem 1.1 The Serre spectral sequence \( \{ E^t_{s,*}(p_{k-r,r}), d^t \} \) of the path fibration \( p_{k-r,r} : \mathcal{P}(\mathbb{F}_{k-r,r}) \to \mathbb{F}_{k-r,r} \) has the following properties:

\[
\begin{align*}
(i) \quad & E^2_{s,*}(p_{k-r,r}) \cong H_*(\mathbb{F}_{k-r,r}) \otimes H_*(\Omega(\mathbb{F}_{k-r,r})), \\
(ii) \quad & E^t_{s,*}(p_{k-r,r}) \cong E^s_{s,*}(p_{k-r,r}), \text{ for } r \leq n, \quad \text{and} \\
(iii) \quad & E^{n+1}_{s,*}(p_{k-r,r}) \cong E^{\infty}_{s,*}(p_{k-r,r}) = \mathbb{Z},
\end{align*}
\]

where homology is with integral coefficients.

To start, consider the following segment of the fundamental fiber sequence \( \mathcal{F}_k \) of \( \mathbb{F}_k(\mathbb{R}^{n+1}) \) (see Chapter II, §1):

\[
\begin{align*}
\cdots \leftarrow \mathbb{F}_{k-1,t+1} \leftarrow \mathbb{F}_{k-t, t} \leftarrow \cdots \\
\downarrow & \downarrow \downarrow \downarrow \\
\cdots \leftarrow \mathbb{R}^{n+1}_{t+1} \leftarrow \mathbb{R}^{n+1}_t \leftarrow \cdots.
\end{align*}
\]

Recall that the vertical maps are the projections on the first nonconstant factor. Observe that

\[ \mathbb{R}^{n+1}_t \cong (\bigvee_{j=1}^t S_{t+1}) \cong (S^n)^{\vee t} \]

for \( r \leq t < k \), where the notation is that of Chapter II (see Proposition 1.1 of Chapter II).

The proof of Theorem 1.1 will be by induction, but first we need some preparatory work. To simplify the notation, put

\[ E = \mathbb{F}_{k-t,t}, \quad B = \mathbb{R}^{n+1}_t, \quad F = \mathbb{F}_{k-t-1,t+1}, \quad p = p_{k-t,t}, \]

and consider the fibration

\[ p : E \to B, \quad (q_1, \ldots, q_t, x_1, \ldots, x_{k-t}) \mapsto x_1, \]

with fiber \( F \). The map \( i : F \to E \) that imbeds \( F \) as the fiber at the basepoint induces the commutative diagram

\[
\begin{array}{ccc}
\Omega(F) & \to & \mathcal{P}(F) \to F \\
\downarrow \Omega(i) & & \downarrow = \\
\Omega(E) & \to & \Omega(E, F) \to F \\
\downarrow = & & \downarrow \\
\Omega(E) & \to & \mathcal{P}(E) \to E
\end{array}
\]

of path fibrations, where \( \Omega(E, F) = \{ (x, \alpha) \mid \alpha(r) = x \in F \} = i^* \mathcal{P}(E) \). The vertical maps are the natural maps induced by \( i \) or by the identity.

Recall that there is a section \( \rho : \mathbb{R}^{n+1}_t \to \mathbb{F}_{k-t,t} \), with restrictions to the spheres \( \{ S_{t+1} \mid 1 \leq j \leq t \} \) that realize the elements \( \{ \alpha_{t+1} \mid 1 \leq s < t + 1 \} \).