5. The Motion of the Earth

As a preparation for later chapters we shall now review the description of the motion of the Earth relative to the fixed stars.

Through a few examples we first demonstrate the application of vector calculus to the analysis of the motion of a rigid body. We then apply these results to the motion of the Earth.

5.1 Examples

Example 5.1. Vectors and the Rotation of a Rigid Body

A rigid body rotates with the constant angular velocity \( \omega \) around a fixed axis \( A-A' \). The angular velocity vector \( \omega \) is defined as a vector along the rotational axis pointing in the direction where the rotation and \( \omega \) defines a right-hand spiral. The magnitude of \( \omega \) is \( |\omega| \). See the figure.

Let \( P \) be some point fixed on the rotating body.

(1) Show that the velocity of \( P \) is

\[ \mathbf{v} = \omega \times \mathbf{r} . \]  

where \( \mathbf{r} \) is a vector from some origin \( O \) on the rotation axis to \( P \).
(2) Show that the acceleration $\mathbf{a}$ of the point $P$ is

$$\mathbf{a} = \omega \times (\omega \times \mathbf{r}) = -\omega^2 \mathbf{p},$$

where $\mathbf{p}$ is the projection of $\mathbf{r}$ on a plane perpendicular to $\omega$. See the figure.

Solution. During the rotation, the point $P$ performs a uniform circular motion with the angular velocity $\omega$ in a plane perpendicular to $\omega$. The radius in that circular motion is $\mathbf{p}$.

(1) The velocity $\mathbf{v}$ of $P$ has the magnitude $|\mathbf{v}| = \omega \mathbf{p}$ and is in the direction along the tangent to the circle. We obtain a vector in this direction by taking the vectorial product (cross product) $\omega \times \mathbf{p}$, and the magnitude of this vector is $|\omega \times \mathbf{p}| = \omega \mathbf{p}$. Thus:

$$\mathbf{v} = \omega \times \mathbf{p}.$$  

(5.3)

Note: when we construct the vector product of two vectors we may translate them to a common origin. The area of the parallelogram spanned by $\omega$ and $\mathbf{p}$ is the same as the area of the parallelogram spanned by $\omega$ and $\mathbf{r}$ (draw the vectors from a common origin). This means that

$$|\omega \times \mathbf{r}| = |\omega \times \mathbf{p}|.$$  

Since the direction of $\omega \times \mathbf{r}$ and $\omega \times \mathbf{p}$ is also the same, we conclude that

$$\mathbf{v} = \omega \times \mathbf{r} = \omega \times \mathbf{p}.$$  

(5.4)

(2) The acceleration of $P$ has the magnitude $a = \omega^2 \mathbf{p}$ (see Example 1.2) and is directed toward the center of the circle, i.e., toward the axis of rotation. Therefore

$$\mathbf{a} = -\omega^2 \mathbf{p}.$$  

(5.5)

Since $\omega$ and $\mathbf{v}$ are mutually perpendicular, we find that the magnitude of $\omega \times \mathbf{v}$ is $|\omega \times \mathbf{v}| = \omega \mathbf{v} = \omega^2 \mathbf{p}$. The vector $\omega \times \mathbf{v}$ then points in the direction of $-\mathbf{p}$. Therefore:

$$\omega \times \mathbf{v} = -\omega^2 \mathbf{p} = \mathbf{a}.$$  

(5.6)

Substituting (5.4) into (5.6) we get:

$$\mathbf{a} = -\omega^2 \mathbf{p} = \omega \times (\omega \times \mathbf{r}).$$  

(5.7)

Alternatively, (5.7) can be seen as a result of direct differentiation of (5.4):

$$\frac{d\mathbf{v}}{dt} = \frac{d}{dt}(\omega \times \mathbf{r}) = \omega \times \frac{d\mathbf{r}}{dt} = \omega \times \mathbf{v} = \omega \times (\omega \times \mathbf{r}).$$