7.2. Boundary-value Problems for Non-homogeneous Equations. We examine the boundary-value problem

\begin{align}
\begin{cases}
P(\partial_x)u(x) = f(x), & x_n > 0, \\
\gamma B_j(\partial_x)u(x') = f_j(x'), & x' \in \mathbb{R}^{n-1}, \quad j = 1, \ldots, l.
\end{cases}
\end{align}

(7.12)

Proposition 7.2. Let \( \alpha \) and the operator \( P \) satisfy the hypotheses of Proposition 7.1, and let the boundary-value problem (1.1), (2.2) be regular in \( U^{(r)}_a \), where \( r \geq \overline{m} \). Then, for any \( f \in U^{(r-1)}_{\overline{m}} \) and \( f_j \in H_{-\infty}(\mathbb{R}^{n-1}) \), the boundary-value problem (7.12) has a unique solution \( u \in U^{(r)}_a \).

Proof. Suppose that \( u \in U^{(r)}_a \) is a solution of the problem (7.12) and that \( v \in U^{(r)}_a \) is a solution of the equation (7.1) constructed in Proposition 7.1. Then \( w = u - v \in U^{(r)}_a \) is a solution of the problem

\begin{align}
\begin{cases}
Pw = 0, & x_n > 0, \\
\gamma B_jw = g_j = f_j - \gamma B_jv, & j = 1, \ldots, l,
\end{cases}
\end{align}

(7.13)

and \( w \) exists and is unique because the problem (1.1), (2.2) is regular in \( U^{(r)}_a \).

Remark 7.1. The hypotheses of Proposition 7.2 are satisfied by

a) the Cauchy problem for the Gårding hyperbolic or for Petrovskij parabolic operator \( P \) provided that \( \alpha > \overline{\alpha} \) and \( r \geq 1 \) (this statement follows from Corollary 3.1);

b) the Dirichlet problem for the elliptic operator \( P(\partial_x) \) of order \( m = 2l \), with \( n \geq 3 \), if \( \overline{\alpha} < \alpha < \overline{\alpha}+1 \) and \( r \geq l - 1 \) (this statement is a consequence of Proposition 3.3).

Chapter 6

Sharp and Diffusion Fronts of Hyperbolic Equations

Hyperbolic equations constitute a large class of partial differential equations. The most well-known representative of this class is the wave equation

\[ \frac{\partial^2 u}{\partial t^2} - k^2 \sum \frac{\partial^2 u}{\partial z_i^2} = 0, \]

which describes the propagation of waves with speed \( k \). In analogy with this example the solutions of arbitrary hyperbolic equations are also referred to as waves. An elementary wave arising from instantaneous perturbation at a point has a singularity on a cone in space-time on the so-called wave front, is analytic outside it and vanishes outside its convex hull. For example, the front of the

\[ \text{This chapter has been written by V.A. Vasil'ev}. \]
wave equation is defined by the equation \( k^2 t^2 = \sum z_i^2 \), \( t > 0 \). The subject matter of this chapter is the study of the qualitative behaviour of the wave as it approaches its front.

The wave equations themselves provide various examples of such qualitative behaviour. Thus in our four-dimensional space-time (and in any other \( 2l \)-dimensional space, \( l \geq 2 \)) the signal is observed only for an instant when it passes the observer. In contrast, in odd-dimensional space, the signal continues to make sound beyond the moment \( t_0 \) of reception (with intensity proportional to \( 1/\sqrt{t^2 - t_0^2} \)). The first situation enables us to communicate by means of sound while the second explains the fact that the "acoustic layer" in an ocean, which is otherwise an excellent conductor of separate signals, is unsuitable for transmitting even slightly complicated information. Both examples of the behaviour of sound waves have analogues for arbitrary hyperbolic equations. In the language of the general theory we say that, in the first case, the internal component of the complement of the front is a lacuna while, in the second case, diffusion of the wave occurs on the side of such a component; the external component is a lacuna for all dimensions (and for all hyperbolic equations).

§ 1. Basic Notions

1.1. Hyperbolic Operators. Consider the linear space \( \mathbb{R}^n \) with coordinates \( x_i \) (\( i = 1, \ldots, n \)) and a differential operator \( P \) with constant coefficients on \( \mathbb{R}^n \), that is, a finite sum of the form

\[
\sum P_\alpha (v - i \partial / \partial x_1)^{\alpha_1} \cdots (v - i \partial / \partial x_n)^{\alpha_n},
\]

where the \( P_\alpha \) are constants enumerated with respect to the multi-index \( \alpha = (\alpha_1, \ldots, \alpha_n) \). To such an operator there corresponds the characteristic polynomial

\[
P = \sum P_\alpha \xi^\alpha, \quad \xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n},
\]