Chapter 3

Jacobians and Abelian Varieties

These varieties play an important role in the theory of algebraic curves, even though — formally speaking — they are outside its one-dimensional scope. In fact, their geometry is no more complicated than that of curves. For instance, the Jacobian of a complex algebraic curve $C$ can be thought of as a complex torus. The lattice is given by the period matrix of regular differentials on $C$ (cf. Example 2 of Sect. 1.3). This torus is algebraic, since it is associated with a (non-singular) algebraic subvariety of $\mathbb{CP}^n$ (cf. Chap. 2, Sect. 1.9). In view of Abel’s theorem, the points on the torus can be identified with the linear equivalence classes of divisors of degree 0 on $C$. This presentation of Jacobians is adapted for applications and holds over any ground field $k$. It is developed in § 2. As we see, a Jacobian has two algebraic structures at once: it is both a variety and a group. This brings us to the subject of algebraic groups and abelian varieties.

§ 1. Abelian Varieties

A typical example is any complex algebraic torus $\mathbb{C}^n/\Lambda$. The main topic of this section is a restatement of the condition that $\mathbb{C}^n/\Lambda$ is algebraic, in terms of the lattice $\Lambda$. This leads to an important additional structure of abelian varieties, known as a polarization. The section concludes with a discussion of one-dimensional abelian varieties, that is, elliptic curves.

1.1. Algebraic Groups. An algebraic group is an algebraic variety $G$, together with a regular multiplication operation $G \times G \to G$, $(g, h) \mapsto g \cdot h$, on its points, and a regular inverse operation $G \to G$, $g \mapsto g^{-1}$. The additive terminology and notation is customary in the commutative case.

Example 1. A finite-dimensional vector space over $k$ is an algebraic group with respect to addition.

Example 2. The group $\text{GL}(n, k)$ of invertible $n \times n$-matrices with elements in $k$ is a multiplicative algebraic group. Similarly, there is a natural algebraic group structure on the group $\text{Aut} \mathbb{P}^n$ of automorphisms of projective space $\mathbb{P}^n$. This group is isomorphic to $\mathbb{PGL}(n+1, k)$.

Remark. In the same sense, the automorphism group $\text{Aut} V$ of any algebraic variety $V$ is algebraic.

Since the action of an algebraic group $G$ on itself is transitive and regular, we obtain the following
Proposition. (The underlying variety of) any algebraic group $G$ is nonsingular.

1.2. Abelian Varieties

Definition 1. A commutative algebraic group $A$ on an irreducible projective variety is called an abelian variety.

Remark 1. As a matter of fact, the commutativity condition is unnecessary. Indeed an algebraic group on an irreducible projective variety is always commutative (see Mumford [1970]).

Remark 2. A regular mapping of abelian varieties which preserves 0 is a homomorphism (cf. Mumford [1970]). So, the group structure of an abelian variety is uniquely determined by specifying 0.

Definition 2. An abelian variety of dimension 1 is an elliptic curve.

Example. Given an elliptic curve $C$, the infinite group of translation automorphisms $q \mapsto q + p$, with $p, q \in C$, acts without fixed points for $p \neq 0$. It follows that $g(C) = 1$. Conversely, if $p$ is a point on a curve $C$ of genus 1, there is a unique elliptic curve structure on $C$ with $0 = p$. The sum $p_3 = p_1 + p_2$, for $p_1, p_2 \in C$, is defined to be the only element of the linear system $|p_1 + p_2 - p|$, which is zero-dimensional by Riemann's formula (cf. Sect. 2.6).

The relative ease with which one obtains many results on abelian varieties (as in the above Remarks) over $k = \mathbb{C}$ (and, by the Lefschetz principle, in characteristic 0) is explained by the following

Theorem. With any complex abelian variety $A$ of dimension $n$, we can associate an $n$-dimensional complex torus.

Indeed, $A^{an}$ is a compact, connected, complex Lie group of dimension $n$ (cf. Chap. 2, Sect. 1.9). It is proved in Lie group theory that any such group is a complex torus $\mathbb{C}^n / \Lambda$, the space $\mathbb{C}^n$ being canonically identified with the tangent space $T$ to $A$ at 0, and the quotient mapping $\mathbb{C}^n \rightarrow \mathbb{C}^n / \Lambda$ with the exponential map $T \rightarrow A^{an}$ (cf. Mumford [1970]). By the comparison theorems (see Serre [1956]), the algebraic variety structure on $A$ can be uniquely reconstructed from the analytic structure of the torus $\mathbb{C}^n / \Lambda$. But by far not every complex torus of dimension $\geq 2$ is algebraic.

1.3. Algebraic Complex Tori; Polarized Tori. A general criterion of when a compact complex manifold is algebraic (more precisely, projective), is provided by Kodaira's theorem (see Griffiths-Harris [1978]). The case of complex tori, though nontrivial, is somewhat simpler (cf. Hartshorne [1981–82] and Mumford [1970]).

Example 1. Let $\mathbb{C}^2 / \Lambda$ be a two-dimensional complex torus, with associated lattice $\Lambda = \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3 + \mathbb{Z}e_4$. As it is homeomorphic to the torus $(\mathbb{R}/\mathbb{Z})^4$, its two-dimensional homology group is generated by the six cycles