Chapter 1

Bifurcations of Equilibria

The theory of bifurcations of dynamical systems describes sudden qualitative changes in the phase portraits of differential equations that occur when parameters are changed continuously and smoothly. Thus, upon loss of stability, a limit cycle may arise from a singular point, and the loss of stability by a limit cycle may give rise to chaos. Such changes are termed bifurcations.

In Chap. 1 and 2 only local bifurcations are investigated, that is, bifurcations of phase portraits near singular points and limit cycles are considered.

In differential equations describing real physical phenomena, singular points and limit cycles are most often found in general position, that is, they are hyperbolic. However, there are special classes of differential equations where matters stand differently. Such classes are, for example, systems having symmetries related to the very nature of the phenomena investigated, and also Hamiltonian systems, reversible systems, and equations that preserve phase volume. Consider, for example, the one-parameter family of dynamical systems on the line with second-order symmetry:

\[ \dot{x} = v(x, \varepsilon), \quad v(-x, \varepsilon) = -v(x, \varepsilon). \]

A typical bifurcation of a symmetric equilibrium in such a system is the pitchfork bifurcation shown in Fig. 1 \((v = x(\varepsilon - x^2))\). In this bifurcation, from the loss of stability by a symmetric equilibrium, two new, less symmetric, equilibria branch out. In this process the symmetric equilibrium position continues to exist, but it loses its stability.

In typical one-parameter families of general (nonsymmetric) systems, pitchfork bifurcations do not occur. Under a small perturbation of the vector field \(v(x, \varepsilon)\) above (although the breaking of symmetry may be ever so slight) the pitchfork in Fig. 1 changes into one of the four pairs of curves in Fig. 2. From these pictures it is evident that the phenomena occurring in response to a smooth, slow change of a parameter in an idealized, strictly symmetric system are qualitatively different from those in a perturbation of it. Therefore, it is necessary to take account of the influence of a slight breaking of symmetry when analysing bifurcations in symmetric systems, if such a break is generally possible. On the other hand, strictly symmetric models occur in some instances. Such is the case, for example, for normal forms (see § 3 below). In these cases it is necessary to investigate bifurcations of symmetric systems within the class of perturbations that do not break symmetry.

The degenerate cases which are avoidable by small generic perturbations of an individual system may become unavoidable when families of systems are studied. Therefore, in the investigation of degenerate cases, instead of studying an individual degenerate equation one should always consider the bifurcations that occur in generic families of systems that display a similar degeneracy in an

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unavoidable form. Technically, this investigation is carried out with the help of the construction of special, so-called versal, deformations; in some sense these contain all possible deformations.

§ 1. Families and Deformations

In this section the transversality theorem and the “reduction principle”, which allows one to lower the dimension of phase space by “neglecting” inessential (hyperbolic) variables, are formulated.

1.1. Families of Vector Fields. We consider a family of differential equations, say,

\[ \dot{x} = v(x, \varepsilon), \quad x \in U \subset \mathbb{R}^n, \quad \varepsilon \in B \subset \mathbb{R}^k. \]

The domain \( U \) is called phase space, \( B \) is called the space of parameters (or the base of the family), and \( v \) is called a family of vector fields on \( U \) with base \( B \). Henceforth, unless stated otherwise, only smooth families will be considered (\( v \) is of class \( C^\infty \)).

1.2. The Space of Jets. Let \( U \) and \( W \) be domains of the real, linear spaces \( \mathbb{R}^n \) and \( \mathbb{R}^m \), respectively. If we choose coordinate systems in \( \mathbb{R}^n \) and \( \mathbb{R}^m \), then the \( k \)-jet of a mapping \( U \to W \) at a point \( x \) is the vector-valued Taylor polynomial at \( x \) with degree \( \leq k \). Similarly, the set of all \( k \)-jets of mappings \( U \to W \) is defined by \( U \times \{ \text{the space of } m \text{-component vector polynomials, of degrees no greater than } k, \text{ in } n \text{ variables, with constant terms in } W \} \), and therefore it is a smooth manifold. The manifold of \( k \)-jets of mappings \( U \to W \) is denoted by \( J^k(U, W) \).