(1962–3). However, problems are not rare in which the smooth and analytic
classifications are completely different. Such, for example, is the simplest problem
on breaking of symmetry, which leads to the classification of pairs of involutions
of the line with a common fixed point. The smooth classification repeats the
formal classification here, whereas there exist formally equivalent, but analytic-
ally different classes depending upon arbitrary holomorphic functions (S.M.
Voronin (1981, 1982) Écalle (1975), Dufour (1977)).

§ 4. The Theory of Bifurcations in the Work of A.A. Andronov

4.1. The Point of View of Function Space. “The main thing we must do with
the differential equations of physical models is to investigate what is possible and
what it is necessary to change in them”, said Poincaré (La valeur de la science,
Part 2, Chapter V (Analysis and physics), p. 222 (1905b)). In 1931, following this
prescription, A.A. Andronov (1933) came out with a vast program (also set forth
in the preface to Andronov and Khajkin, 1937) which differs from the contempo-
rary program of catastrophe theorists only in that the qualitative theory of
differential equations and Poincaré’s theory of bifurcations take the place of
Whitney’s theory of singularities of differentiable mappings, as yet uncreated at
his time.

From the mathematical point of view, the basis of Andronov’s approach
was to consider a whole class of admissible models of the phenomenon in-
vestigated at the same time, instead of a single model. That class, in fact, is usually
a function space, since the functions that enter into the right-hand sides of the
equations (for example, the characteristics of the nonlinear elements of an electric
circuit) are usually known only approximately. In some cases the function space
comes down to a finite-dimensional space of parameters, but here also the exact
values of the parameters are usually unknown. It is above all a question of
studying systems which are typical (in the given class of models) and parameter
values corresponding to typical systems.

4.2. Structural Stability. A.A. Andronov reasons in the following way. Since
the exact values of the parameters in a model are unknown, a conclusion drawn
from the mathematical investigation of a model is worth the confidence of the
practical worker only to the extent that this conclusion is stable with respect to
small changes of the parameters. For example, a conclusion about the periodicity
of a steady-state regime of motion of a system is well founded only if the
corresponding periodic solution of the equations of the model is preserved under
all small changes in the model (within the corresponding function space); here,
of course, small changes in the periodic solution itself and in its period are
allowed.

In formalizing these ideas, A.A. Andronov arrived at the concept of structural
stability (in his terminology roughness) of properties of the system being in-
vestigated; the class of objects equivalent to the given one in relation to the
properties of interest to us must be open in the corresponding function space of objects.

Andronov applied these ideas, above all, to dynamical systems and to orbital topological equivalence. In the 2-dimensional case, on the plane or on the sphere, he, together with L.S. Pontryagin, succeeded in proving the structural stability of generic systems (Andronov and Pontryagin (1937)). However, Andronov's program itself was more general and concerned any classification of arbitrary objects of analysis, for example, functions, fields, bifurcations, etc., for which he at once gave examples (see Andronov (1933), Andronov and Khajkin (1937)).

Depending on the classification being studied, the partition of the function space into classes can be very complex. For example, it can be discrete in some regions of the function space or parameter space and continuous in others; see Fig. 14. Indeed, this is how things stand for the partition of the space of dynamical systems with a more than 2-dimensional phase space into topological orbital equivalence classes; this was cleared up only in the sixties, principally thanks to Smale (1966).

![Fig. 14. Partition into classes](image)

One should emphasize that the concept of roughness (structural stability) appears in Andronov's work as both a general physical and a general mathematical idea. The objects being studied need not necessarily be dynamical systems, and the classification need not necessarily be topological. In Andronov's terminology, the classification we use is determined by the questions we ask about a system, and the requirement of structural stability of models forbids our asking questions that are too precise on the qualitative behaviour of systems in those cases when a small change of the model changes the answers to these questions. This means a useful "qualitative" classification partitions the function space into discrete (and not continuous) parts.

I note such a classification has now been found in the theory of singularities of differentiable mappings (thanks to the efforts of Thom (1964), A.N. Varchenko (1975, 1974), Mather (1973), Looijenga (1974), Wirthmüller (see Gibson et al. (1977), and others), but is unknown in the theory of dynamical systems.

### 4.3. Bifurcation Sets.

According to the arguments set forth above, nongeneric objects need not necessarily be met, and one may neglect them in an initial analysis (for example, the singular points of a generic vector field are nodes, foci and saddles, but not centers at all; and therefore the investigation of centers may be put off as less important).