Chapter XIX. The Linearised Navier–Stokes Equations

Introduction

The object of this chapter is to present a certain number of results on the linearised Navier–Stokes equations. The Navier–Stokes equations, which describe the motion of a viscous, incompressible fluid were introduced already, from the physical point of view, in §1 of Chap. IA. These equations are nonlinear. We study here the equations that emerge on linearisation from the solution $(u = 0, p = 0)$. This is an interesting exercise in its own right. It corresponds to the case of a very slow flow, and also prepares the way for the study of the complete Navier–Stokes equations.

This Chap. XIX is made up of two parts, devoted respectively to linearised stationary equations (or Stokes’ problem), and to linearised evolution equations. Questions of existence, uniqueness, and regularity of solutions are considered from the variational point of view, making use of general results proved elsewhere. The functional spaces introduced for this purpose are themselves of interest and are therefore studied comprehensively.

§1. The Stationary Navier–Stokes Equations: The Linear Case

Orientation: Presentation of Stationary Problem

The motion of a homogeneous, viscous, incompressible fluid is described by the Navier–Stokes equations. These have been introduced in §1 of Chap. IA. Here, we shall study the Navier–Stokes equations in the case of a stationary (or permanent) flow after linearisation. It is therefore the Stokes problem which will be considered, i.e.: determine the velocity $u = (u_1, \ldots, u_n)$\(^{(1)}\) and the pressure $p$, in a domain $\Omega$, such that:

1. \[ - \nu \Delta u + \text{grad} \, p = f \quad \text{in} \quad \Omega, \quad \Omega \subset \mathbb{R}^n(v > 0) \]
2. \[ \text{div} \, u = 0 \quad \text{in} \quad \Omega, \]
3. \[ u = 0 \quad \text{over} \quad \partial \Omega. \]

\(^{(1)}\) Also denoted by $U$ in Chap. IA, §1.

\(^{(2)}\) We take the space $\mathbb{R}^n$ in order to treat simultaneously the usual cases $n = 2$ and $n = 3$. 

The boundary conditions, \( u = 0 \) on \( \partial \Omega \), correspond to the case of a rigid boundary \( \partial \Omega \) with the fluid adhering to it. The approach to problem (1)–(3) will be **variational**. Some function spaces adapted to the problem will be introduced (Sect. 1), which will allow us to pose a weak form equivalent to (1), (2), (3). The existence and uniqueness (Sect. 2) of the solution then follows from the Lax–Milgram theorem\(^\text{(3)}\). After a study for bounded \( \Omega \), these results will be extended to cases of problems which are totally nonhomogeneous, with unbounded domain \( \Omega \) and to the study of eigenvalue problems. We also demonstrate the interest of the concept of penalisation. The delicate question of regularity in broached in Sect. 3.

### 1. Functional Spaces

#### 1.1. Notation

Let \( \Omega \) be an open set of the Euclidean space \( \mathbb{R}^n \). In what follows two types of regularity hypotheses will be used on \( \Omega \). A strong hypothesis:

\begin{align}
\text{1.1} \quad & \begin{cases}
\text{the boundary } \Gamma \text{ of } \Omega \text{ is a manifold of dimension } (n - 1) \text{ of class } \mathcal{C}^r \text{ (} r \text{ precisely) and } \Omega \text{ is locally situated on one side of } \Gamma.
\end{cases}
\end{align}

We say that \( \Omega \) satisfying (1.1) is of class \( \mathcal{C}^r \). The second hypothesis is less restrictive (if \( r \geq 1 \)):

\begin{align}
\text{1.2} \quad & \begin{cases}
\text{the boundary } \Gamma \text{ of } \Omega \text{ is locally Lipschitz, i.e. is represented locally by the graph of a Lipschitz function}^{\text{(4)}}.
\end{cases}
\end{align}

If \( \Omega \) satisfies (1.2), it is then locally star-shaped, i.e. every point \( x \) of \( \Gamma \) has a neighbourhood \( \mathcal{O} \) such that \( \mathcal{O} \cap \Omega \) is star-shaped with respect to one of its points (Nečas [1], [2]). Our definition of a star-shaped open set is a little more restrictive than the usual definition: we say that \( \mathcal{O} \) is star-shaped with respect to the origin if \( \lambda \overline{\mathcal{O}} \subset \mathcal{O} \) for all \( \lambda > 0 \).

We further assume that all of the open sets \( \Omega \) of this Chap. XIX are **connected**. With every open set \( \Omega \) we associate the usual spaces \( \mathcal{D}(\Omega) \) (and \( \mathcal{D}(\overline{\Omega}) \)), \( L^p(\Omega) \), and the Sobolev spaces \( W^{m,2}(\Omega) \) (\( m \) an integer and \( 1 \leq p \leq +\infty \)), all equipped with their usual norms. If \( m = 2 \), we set \( W^{m,2}(\Omega) = H^m(\Omega) \). The closure of \( \mathcal{D}(\Omega) \) in \( W^{m,2}(\Omega) \) will be denoted \( W^{m,2}_0(\Omega) \) \( (H^m_0(\Omega) \text{ if } m = 2) \). For convenience, due to the frequent use of vector-valued functions with \( n \) components in what follows, we shall use the following notation in this chapter:

\[
\mathcal{D}(\Omega) = [\mathcal{D}(\Omega)]^n, \quad L^s(\Omega) = [L^s(\Omega)]^n, \\
W^{m,2}(\Omega) = [W^{m,2}(\Omega)]^n, \quad H^m(\Omega) = [H^m(\Omega)]^n.
\]

All of these spaces (with the exception of \( \mathcal{D}(\Omega) \)) are equipped with the natural

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\((3)\) See Chap. VI and VII.

\((4)\) We then say that \( \Omega \) is an open Lipschitz set.