Chapter 1
Critical Points of Functions

One of the most thoroughly studied branches of the theory of singularities is the investigation and classification of degeneracies of critical points of functions. Generic functions have only nondegenerate critical points. More complex singularities vanish under small perturbations, decomposing into nondegenerate ones.

However, in families of functions that depend on several parameters, degenerate critical points may occur in an irremovable manner. For example, the family of functions \( x^3 + \lambda x \) has for the value \( \lambda = 0 \) of the parameter a degenerate critical point; any close one-parameter family has, for a close value of the parameter, a degeneracy of the same kind. As the number of parameters increases, in families of functions there arise degeneracies of ever increasing complexity.

In this chapter we describe the initial segment of the classification of critical points of functions. This classification of the simplest degeneracies of critical points turned out to be closely related to the classification of simple Lie groups, of reflection groups, and of braid groups.

To simplify the exposition, we restrict ourselves mainly to the case of holomorphic functions, diffeomorphisms, and so on. The theory carries over, with practically no modifications, to the case of real smooth functions; some differences arising in the real case will be pointed out. The classification of real critical points is given in subsection 2.8.

§ 1. Invariants of Critical Points

Here we give the basic definitions concerning critical points of functions.

1.1. Degenerate and Nondegenerate Critical Points

Definition. A point is said to be a critical point of a smooth function \( f \) if at that point the derivative of \( f \) is equal to zero.

The value that the function takes at a critical point is called a critical value.

Example. The function \( f(x) = x^3 - \lambda x \) of the variable \( x \in \mathbb{C} \) has for each \( \lambda \neq 0 \) the two critical points \( \pm\sqrt[3]{\lambda/3} \) with respective critical values \( \mp\sqrt[3]{4\lambda^3/27} \). For \( \lambda = 0 \) these two critical points "merge" into a single critical point 0.

The critical points of functions are divided into generic (general-position, or nondegenerate) critical points and degenerate critical points.
Invariants of Critical Points

Definition. A critical point is said to be nondegenerate (or a Morse critical point) if the second differential of the function at that point is a nondegenerate quadratic form.

Example. The function \( f(x) = x^3 - \lambda x \) has for \( \lambda \neq 0 \) the pair of nondegenerate critical points \( \pm \sqrt{\lambda/3} \) and for \( \lambda = 0 \) the degenerate critical point 0 (Fig. 1).

The degree of degeneracy of the second differential is the simplest indicator of how degenerate a critical point is.

Definition. The corank of a critical point of a function is the dimension of the kernel of its second differential at the critical point.

Examples. The corank of any Morse critical point is equal to zero. The corank of the critical point 0 of the function \( f = x_1^3 + x_2^3 + \cdots + x_n^3 \) is equal to one.

1.2. Equivalence of Critical Points. Let us consider the set \( \mathcal{O}_n \) of function-germs at the point \( 0 \in \mathbb{C}^n \).

Definition. Two function-germs at zero are said to be equivalent if one is taken into the other by a biholomorphic change of coordinates that keeps the point zero fixed.

This notion of equivalence can be alternatively described as follows. Let \( \mathcal{D}_n \) denote the group of germs of biholomorphic maps \( g: (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0) \). This group acts on the space \( \mathcal{O}_n \) of function-germs by the rule \( g(f) = f \circ g^{-1} \), where \( f \in \mathcal{O}_n, \ g \in \mathcal{D}_n \). The orbits of this action are exactly the equivalence classes of function-germs.

Definition. Two critical points are said to be equivalent if the function-germs that define them are equivalent. The equivalence class of a function-germ at a critical point is called a singularity.