vector fields on the space of quasihomogeneous polynomials of degree \( N = \deg f_0 \) is trivial (i.e., equal to \( \{0\} \)).

In other words, \( f \) satisfies Condition \( B \) if \( S_0^1 = 0 \). Thus, Condition \( B \) is actually a condition only on \( f_0 \).

**Theorem BT.** If \( f \) satisfies Condition \( B \), then Theorem \( T_{r,p} \) holds for \( r = p \geq 1 \).

**Definition.** The negative Lie algebra \( g^- \) of type \( v \) is defined to be the algebra of vector fields of the form \( \sum v_i \delta_i \) such that all monomials of each of the polynomials \( v_i \) are of degree strictly smaller than the degree of the monomial \( x_i \) (i.e., than \( v_i \)).

Note that \( g^- \) is a finite-dimensional Lie algebra.

**Definition.** The series \( f = f_0 + f_1 + \cdots \) is said to satisfy Condition \( C \) if the isotropy algebra of the point \( f_0 \) under the action of the negative Lie algebra \( g^- \) on the space of polynomials of degree no higher than \( N = \deg f_0 \) is trivial (i.e., equal to \( \{0\} \)).

Note that Condition \( C \), too, is imposed only on \( f_0 \).

**Theorem CT.** If \( f \) satisfies Condition \( C \), then \( I_f^+ = A \cap I_f \).

**Corollary.** Let \( f \) satisfy Condition \( C \) and let \( e_1, e_2, \ldots \) be quasihomogeneous polynomials of all possible degrees \( N + p, p \geq 0 \), whose images under the natural maps \( \mathcal{A}_p \to A_p^\infty \) form bases in the spaces \( A_p^\infty \) of the spectral sequence. Then the images of \( e_1, e_2, \ldots \) in the local algebra \( Q_f = A/I_f \) are \( \mathbb{C} \)-linearly independent.

In other words, the tangent space of the deformation \( f + \sum \lambda_i e_i \) intersects the tangent space to the orbit of \( f \) at a single point.

## Chapter 2

**Monodromy Groups of Critical Points**

Morse theory studies the restructurings, perestroikas, or metamorphoses that the level set \( f^{-1}(x) \) of a real function \( f: M \to \mathbb{R} \), defined on a manifold \( M \), undergoes as \( x \) passes through the critical values of \( f \). The Picard-Lefschetz theory is the complex analogue of Morse theory. In the complex case the set of critical values does not divide the range \( \mathbb{C} \) of a complex-valued function into connected components, and no restructurings occur: all level manifolds close to a critical one are topologically identical. For this reason, in the complex case, rather than passing through a critical value, one has to go around it in the plane \( \mathbb{C} \) where the function takes its values.

If we fix a small circle that goes around the critical value, then to each point of the circle there corresponds a nonsingular level manifold of the function. The set of all such levels is a fibration over the circle. Going around the circle defines
§1. The Picard-Lefschetz Theory

Here we define the monodromy groups and the related notions of vanishing cycles and Dynkin diagrams, and then we describe the Picard-Lefschetz formulas.

1.1. Topology of the Nonsingular Level Manifold. Consider a holomorphic function $f: \mathbb{C}^n \to \mathbb{C}$, $z \mapsto t$, which has an isolated critical point $z = a$ with critical value $f(a) = \alpha$. Pick a sufficiently small ball $U \subset \mathbb{C}^n$ centered at $a$ such that $U$ contains no other critical points of $f$. Then the level set $f^{-1}(\alpha)$ is an $(n-1)$-dimensional complex manifold, nonsingular everywhere in $U$ except for the point $a$.

Theorem ([247]). The manifold $f^{-1}(\alpha)$ is transverse to the boundary $\partial U$ of the ball $U$ for all sufficiently small values of the radius of $U$.

Fix $U$ such that the assertion of the theorem holds true for $U$ and for all balls centered at $a$ and contained in $U$ and choose a sufficiently small neighborhood $T \subset \mathbb{C}$ of the critical value $\alpha$, such that for every $t \in T$, $t \neq \alpha$, the level manifold $f^{-1}(t)$ is nonsingular inside $U$ and transverse to $\partial U$.

In this way there arises inside $U$ a family $J_t = f^{-1}(t) \cap U$, $t \in T$, of complex hypersurfaces with boundary $\partial J_t = J_t \cap \partial U$.

Definition. The set $J_t = f^{-1}(t) \cap U$, $t \neq \alpha$, is called a nonsingular level set of $f$ near the critical point $a$.

The family of hypersurfaces $J_t$, $t \in T$, fills the domain $V_T = f^{-1}(T) \cap U$. Let $T'$ and $V_T'$ denote the punctured neighborhood $T \setminus \{\alpha\}$ of the critical value $\alpha$ and its preimage $f^{-1}(T') = V_T \setminus V_a$, respectively. The function $f$ induces maps $V_T \to T$ and $V_T' \to T'$ (Fig. 12)

It follows from the implicit function theorem that the maps $f: V_T' \to T'$ and $f|_{V_T}: \partial V_T \to T$, where $\partial V_T = V_T \cap \partial U$, are locally trivial fibrations.

Remark. Since the disk $T$ is contractible, the fibration $\partial V_T \to T$ is trivial. Moreover, the direct product structure in this fibration is unique up to homotopy.

Let $\mu$ be the multiplicity of the critical point $a$. The topology of the nonsingular level manifold $V_t$ is described by the following theorem.

Theorem ([247]). $V_t$ is homotopy equivalent to a wedge (bouquet) of $(n-1)$-dimensional spheres.