Chapter 2
Class Field Theory

(Main references: Cassels, Fröhlich (1967), Artin, Tate (1968), Neukirch (1986))

There are two main problems in the theory of algebraic number fields: On the one hand the description of the arithmetical properties of a given number field and on the other hand the description of number fields with given arithmetical properties.

The theory of complex function fields in one variable delivers us an example of the resolution of the second problem in the case of function fields: The finite extensions $L$ of $\mathbb{C}(z)$ which are unramified outside a set $S$ of points of the Riemann surface $F = \mathbb{C} \cup \{ \infty \}$ of $\mathbb{C}(z)$ correspond to the subgroups $U$ of finite index of the fundamental group of the topological space $F - S$. Thereby the covering space of $F$ given by $U$ corresponds to the Riemann surface of $L$ (see also Chap. 3, Example 13). Unfortunately up to now we have no result in the theory of number fields of this generality. We can express this saying that up to now we have not sufficiently good understood the notion of the fundamental group in the case of number fields.

Before we come to a more detailed discussion of the two main problems we are going to reformulate them for the purposes of this chapter. With respect to arithmetical properties we ask more concretely for the decomposition behavior of prime ideals in finite extensions and since every finite extension is contained in a normal extension we restrict to normal extensions and describe the arithmetical properties of the extensions (and its subextensions) by means of the decomposition and ramification groups (Chap. 1.3.7).

In Chap. 1.3.9 we have seen that we have a very detailed description of the arithmetical properties of the cyclotomic field $\mathbb{Q}(\zeta_p)$: The Galois group of $\mathbb{Q}(\zeta_p)$ over $\mathbb{Q}$ is canonically isomorphic to the group $(\mathbb{Z}/p\mathbb{Z})^\times$. For a prime $q \neq p$ the Frobenius automorphism $F_q$ is given by $F_q(\zeta_p) = \zeta_p^q$, and the $j$-th ramification group of $p$ in the upper numeration corresponds by this isomorphism to the subgroup $\{ a \in (\mathbb{Z}/p\mathbb{Z})^\times | a \equiv 1 \pmod{p^j} \}$.

It turns out that such a beautiful description of the arithmetical properties is possible for any abelian extension $L/K$ of number fields (an abelian extension $L/K$ is a normal extension $L$ of $K$ with abelian Galois group). The corresponding theory is called class field theory. It is the subject of this chapter.

According to the local-global principle (Chap. 1.4), class field theory begins with the study of local fields $K$. The abelian extensions of $K$ correspond to the closed subgroups of finite index of the multiplicative group $K^\times$ of $K$ (§ 1.3). For global fields $K$ the idele class group (Chap. 1.5.5) plays the role of the multiplicative group in the local case: The abelian extensions of the global field $K$ correspond to the closed subgroups of finite index of the idele class group of $K$ (§ 1.5). By the theorem of Bauer (Theorem 1.118) a finite normal extension $L$ of $K$ is characterized by the set of prime ideals which split completely in $L$. For abelian
extensions $L/K$ class field theory delivers a beautiful description of such sets of prime ideals in terms of the subgroup of the idele class group corresponding to $L$ (§1.6).

Beside the two main problems the most interesting problems in class field theory are the explicit construction of class fields and of reciprocity laws. Both problems played a decisive role in the development of the theory. Explicit constructions of abelian extensions are known for local fields (Lubin-Tate extensions (Chap. 1.4.10)), for $\mathbb{Q}$ (§1.1), and for imaginary-quadratic fields (§2). For other global fields one has only partial results. Explicit reciprocity laws are the direct generalization of the quadratic reciprocity law. They allow (in principle) to decide whether an integral number in a field $K$ is an $n$-th power residue with respect to a prime ideal. For a smooth formulation of the results it is necessary to assume that $K$ contains the $n$-th roots of unity. Thus Gauss studied the field $\mathbb{Q}(\sqrt{-1})$ for his formulation of the explicit biquadratic reciprocity law.

The notion of class field was already present in the mind of Kronecker who knew that every abelian extension of $\mathbb{Q}$ is contained in a cyclotomic field and it was his "liebster Jugendtraum" that analogously every abelian extension of an imaginary-quadratic number field can be obtained by "singular moduli" connected with modular functions and elliptic functions (§2). The general concepts of class field theory were developed at the end of the last century by Hilbert and Weber. Hilbert wrote in the introduction to his famous Zahlkörperbericht (Hilbert (1895), p. VII):


In the definition of Weber (1891), (1898) the class field $L$ of an algebraic number field $K$ with respect to a subgroup $H/R_m$ of the ray class group $\mathfrak{a}_m/R_m$ (Chap. 1.5.5) is the (uniquely determined (Chap. 1.6.8)) normal extension of $K$ such that a prime ideal $p$ of $K$ with $p \nmid m$ splits completely in $L/K$ if and only if it belongs to $H$. The Galois group of $L/K$ is isomorphic to $\mathfrak{a}_m/H$. The existence of the class field for a given group $H$ was proved by Takagi (1920) and Artin (1927) showed that there is a canonical isomorphism of $\mathfrak{a}_m/H$ onto $G(L/K)$ given by the Frobenius symbol (Chap. 1.3.7). This isomorphism can be considered as a far reaching generalization of the quadratic reciprocity law. Artin's result completed the foundation of class field theory.

At that time the proofs of the main theorems were extremely complicated (see Hasse (1970)) and the following period is characterized by a simplification and rebuilding of the whole theory. In particular Hasse (1930) developed the local