Approximating Stable Sets Using the \( \vartheta \)-function and Cutting Planes

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Abstract. We investigate an approximation algorithm for the maximum stable set problem based on the Lovász number \( \vartheta(G) \) as an initial upper bound. We strengthen this relaxation by adding two classes of cutting planes, odd circuit and triangle inequalities. We present computational results using this tighter model on several classes of graphs.

1 Linear and Semidefinite Relaxations of the Stable Set Problem

Let \( G = (V, E) \) denote an undirected graph with node set \( V \) and edges set \( E \), where \(|V| = n \) and \(|E| = m\). We denote an edge \( e \) with endnodes \( i \) and \( j \) by \((ij)\).

A stable set \( S \) in \( G \) is by definition a subset \( S \) of pairwise nonadjacent nodes. The maximum stable set problem is the problem of finding a stable set of maximum cardinality. Its cardinality is denoted by \( \alpha(G) \).

An assignment of colors to the vertices of \( G \), one color to each vertex, so that adjacent vertices receive different colors, is called a coloring of \( G \). A \( k \)-coloring of \( G \) is a coloring of \( G \) using \( k \) colors. The chromatic number \( \chi(G) \) of a graph \( G \) is the minimum value \( k \) for which a \( k \)-coloring of \( G \) exists. The complement of a graph \( G \) is denoted by \( \bar{G} \).

It is NP-complete to compute \( \alpha(G) \) or \( \chi(G) \). The Lovász number \( \vartheta(G) \) bounds \( \alpha(G) \) from above and \( \chi(G) \) from below, \( \alpha(G) \leq \vartheta(G) \leq \chi(G) \). This number is also called theta function (\( \vartheta \)-function) of \( G \). \( \vartheta(G) \) can be computed in polynomial time, see [2]. In this work we are primarily interested in estimating \( \alpha(G) \).

The stable set polytope of \( G \) is defined by

\[
STAB(G) := \text{conv}\{ x^S \in \mathbb{R}^n | S \subseteq V \text{ stable set} \},
\]

where \( x^S \) denotes the incidence vector of \( S \), i.e., \( x^S_i = \begin{cases} 1 & \text{for } i \in S \\ 0 & \text{otherwise} \end{cases} \). Thus the maximum stable set problem (STAB) is given by

\[
\text{(STAB)} \quad \max \sum_{i=1}^{n} x_i \quad \text{such that } x \in STAB(G).
\]

A complete description of \( STAB(G) \) by linear inequalities is unlikely to be tractable, in view of the NP-completeness to compute \( \alpha(G) \). The following sets of linear inequalities contain a partial description of \( STAB(G) \).

- **nonnegativity constraints:** \( x_i \geq 0 \quad \forall i \in V \) \hspace{1cm} (1)
- **edge constraints:** \( x_i + x_j \leq 1 \quad \forall (ij) \in E \) \hspace{1cm} (2)
- \( x_i \leq 1 \quad \text{for isolated nodes} \)

Obviously (1) and (2) hold for \( x \in STAB(G) \). Moreover, the integral solutions \( x \) of (1) and (2) are the incidence vectors of stable sets in \( G \). It is well known that (1) and (2) are enough to describe \( STAB(G) \) if and only if \( G \) is bipartite and has no isolated nodes, see [4].

However, if \( G \) contains odd circuits this polytope is no longer enough to describe \( STAB(G) \). Take for instance a triangle \( T \) and set \( x_i = \frac{1}{2} \) for \( i \in V(T) \). Clearly \( x \) satisfies (1) and (2) but is not in \( STAB(G) \).

Therefore the set of inequalities (1) and (2) is extended by adding the class of odd circuit constraints (3),

\[
\sum_{i \in V(C)} x_i \leq \frac{|V(C)| - 1}{2} \quad \text{for } C \text{ odd circuit},
\]

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yielding the polytope $CSTAB(G)$:

$$CSTAB(G) := \{ x \in \mathbb{R}^n | x \text{ satisfies (1), (2) and (3)} \}.$$ 

The polytope $CSTAB(G)$ is still only a (linear) relaxation of $STAB(G)$, i.e. $STAB(G) \subseteq CSTAB(G)$. Any linear function can be optimized over $CSTAB(G)$ in polynomial time [3]. The relaxation $CSTAB(G)$ gives $\alpha(G)$ for the class of $t$-perfect graphs, see [3, 4]. We denote by $S(n)$ the space of $n \times n$ symmetric matrices. A nonlinear relaxation of $STAB(G)$ was introduced by Lovász, see [10], and can also be found in [11]. There are many equivalent ways to define this relaxation, we use the following definition, see [11]:

$$TH(G) := \{ d \in \mathbb{R}^n | \exists X \in S(n) \text{ such that } d = \text{diag}(X), \text{ and } X \text{ satisfies (4)} \}$$

The set $TH(G)$ is a relaxation of the stable set polytope, because the characteristic vector $X$ of a stable set produces the matrix $XX^T$, which is feasible for (4).

One can optimize over $TH(G)$ in polynomial time, see [2]. In fact, see [11, 4, 9]

$$\vartheta(G) := \max \left\{ \sum_i x_i | x \in TH(G) \right\}.$$ 

Our main goal is to get tighter bounds for $\alpha(G)$, starting from $\vartheta(G)$. Therefore we extend this relaxation by adding further constraints to the set $TH(G)$. As already suggested in [7], one can include the triangle inequalities, $X \in MET$ exactly if

$$0 \leq X_{ij} \leq X_{ii}, \quad X_{ii} + X_{jj} - X_{ij} \leq 1,$$

$$-X_{kk} - X_{ij} + X_{ik} + X_{jk} \leq 0,$$

$$X_{kk} + X_{ii} + X_{jj} - X_{ij} - X_{ik} - X_{jk} \leq 1, \quad \text{for } 1 \leq i, j, k \leq n.$$ 

A further refinement can be achieved by intersecting $TH(G)$ with $CSTAB(G)$. Finally we can combine $TH(G)$ with the odd circuit constraints and the triangle inequalities. In a slight abuse of notation we write $X \in CSTAB(G), (TH(G))$ for $\text{diag}(X) \in CSTAB(G), (TH(G))$, respectively.

The remaining sections are organized as follows. In section 2 we summarize the relaxation models. Implementation details are described in section 3. The computational tests appear in section 4. We conclude with comments in section 5.

## 2 Strengthening $\vartheta(G)$ by Linear Cutting Planes

We first rewrite the problem defining $\vartheta(G)$. We write trace$(X)$ to denote the trace of a square matrix $X$. The inner product in $S(n)$ is defined by $\langle A, B \rangle := \text{trace}(AB)$. The symbol $\succeq$ denotes the Löwner partial order, i.e., $A \succeq B$ if $A - B$ is positive semidefinite. $A \succ 0$ means $A$ is positive definite.

We now express the constraints from (4) through $\hat{X} \in S(n + 1)$. Let $E_{ij} := e_i e_j^T + e_j e_i^T$ where $e_i$ stands for the ith unit vector in $\mathbb{R}^{n+1}$. Then $(E_{ij}, \hat{X}) = 0$ corresponds to $\hat{X}_{ij} = 0$. $\text{diag}(X) = d$ is equivalent to $\hat{X}_{ii} = \frac{1}{2} \hat{X}_{i,n+1} + \frac{1}{2} \hat{X}_{n+1,i}$ for $1 \leq i \leq n$. Setting $a_i := e_i e_i^T - \frac{1}{2} e_{n+1} e_i^T$, and using $\hat{X}_{n+1,n+1} = 1$, this is the same as $\langle a_i a_i^T, \hat{X} \rangle = \frac{1}{4}$. The relaxation (5) can therefore be written as the (primal) semidefinite program (PSDP $\vartheta$).

$$(PSDP \vartheta) \quad \vartheta(G) := \max \text{ trace } \hat{X} - 1 \quad \text{s.t.} \quad \langle E_{ij}, \hat{X} \rangle = 0 \quad \forall (ij) \in E(G)$$

$$\hat{X}_{n+1,n+1} = 1$$

$$\langle a_i a_i^T, \hat{X} \rangle = \frac{1}{4} \quad 1 \leq i \leq n$$

$$\hat{X} \in S(n + 1), \hat{X} \succeq 0.$$ 

(7)