7. Maxwell’s Equations

We have analyzed, up to this point, the origin and the consequences of four physical laws associated with static electromagnetic phenomena. The first of them, Gauss’s law, relates the electric field to its sources. In differential form it is written as

\[ \nabla \cdot \mathbf{E} = 4\pi \rho. \quad (7.1) \]

Within the area of application of electromagnetism, this law is of general validity.

In static situations, the electric field also satisfies

\[ \nabla \times \mathbf{E} = 0. \quad (7.2) \]

As we show in the next section, this equation must be modified in the presence of time dependent magnetic fields, giving rise to Faraday’s law, which describes induction phenomena.

The experimental observation of the non-existence of magnetic monopoles establishes that the magnetic field satisfies

\[ \nabla \cdot \mathbf{B} = 0. \quad (7.3) \]

This relationship, like Gauss’s law, has general validity.

Finally, Ampère’s law links the magnetostatic field to the currents originating it, according to

\[ \nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{j}. \quad (7.4) \]

We shall however show that Ampère’s law violates the conservation of charge, and that (7.4) has to be modified to preserve this important physical principle. This correction was introduced by J.C. Maxwell who in such a way unified in a single consistent theory the description of electromagnetic phenomena.

7.1 Time-Dependent Magnetic Fields. Faraday’s Law

In 1831 Michael Faraday and Joseph Henry (1797–1878) almost simultaneously discovered a new phenomenon associated with electric and magnetic fields. This discovery was to be of fundamental importance in technological
applications. This is the phenomenon of magnetic induction, which implies that the time variation of a magnetic field gives rise to electric fields.

More precisely, Faraday’s law indicates that for a closed circuit $C$ in the presence of a varying magnetic field an electromotive force $\mathcal{E}$ is induced (Fig. 7.1). This electromotive force is proportional to the time derivative of the magnetic flux $\Phi$ through the circuit:

$$\mathcal{E} = -\frac{1}{c} \frac{d\Phi}{dt}. \tag{7.5}$$

Here $\Phi$ is the magnetic flux introduced in the previous chapter. The unit for the electromotive force in the cgs system is the statvolt, the same as for the potential $\phi$. In the SI system the unit is the volt.

![Fig. 7.1. Illustration of Faraday’s law](image)

The electromotive force generated in the circuit can be written as the line integral of the electric field along $C$. In integral form, Faraday’s law is then

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{1}{c} \int_S \partial_t \mathbf{B} \cdot d\mathbf{s}, \tag{7.6}$$

where in order to introduce the time derivative into the surface integral we have assumed that the curve $C$ does not vary in time.

Since Faraday’s law is valid for any closed curve – whether it coincides with a material circuit or not – it is possible to find a differential form for it by converting the line integral of $\mathbf{E}$ into a surface integral of its curl. The differential version of Faraday’s law is

$$\nabla \times \mathbf{E} + \frac{1}{c} \partial_t \mathbf{B} = 0. \tag{7.7}$$

Comparing with Maxwell’s equations (4.1), we notice that Faraday’s law is one of the fundamental laws of electromagnetism. In fact, it modifies the