5

Taylor's Expansion

5.1 Introduction: Power Series

It was a great triumph in the early years of Calculus when Newton and others discovered that many known functions could be expressed as "polynomials of infinite order" or "power series," with coefficients formed by elegant transparent laws. The geometrical series for \( \frac{1}{1 - x} \) or \( \frac{1}{1 + x^2} \)

\[
(1) \quad \frac{1}{1 - x} = 1 + x + x^2 + \cdots + x^n + \cdots
\]

\[
(1a) \quad \frac{1}{1 + x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots
\]

valid for the open interval \(|x| < 1\), are prototypes (see Chapter 1, p. 67).

Similar expansions of the form

\[
f(x) = a_0 + a_1x + \cdots + a_nx^n + \cdots = \sum_{v=0}^{\infty} a_v x^v,
\]

with numerical coefficients \(a_v\), will be derived in this chapter for many other functions.

The following are striking examples:

\[
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots;
\]

\[
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n + 1)!} + \cdots;
\]

\[
\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots
\]

These series expansions are valid for all \(x\).
Newton’s General Binomial Theorem. The expansion
\[(1 + x)^a = 1 + \frac{a}{1!}x + \frac{a(a-1)}{2!}x^2 + \cdots \]
\[= \sum_{v=0}^{\infty} \binom{a}{v} x^v\]
is valid for \(|x| < 1\) and any exponent \(a\).

To explain the precise meaning of such expansions, we consider the polynomial of order \(n\) formed as the sum of the first \(n + 1\) terms of the series, the \(n\)th “partial sum,”
\[S_n = \sum_{v=0}^{n} a_v x^v.\]
The formula
\[f(x) = \sum_{v=0}^{\infty} a_v x^v, \quad \text{for } |x| < a\]
then means: For \(n \to \infty\) the sequence \(S_n\) tends to the value of the function \(f(x)\) at each point \(x\) in the interval \(|x| < a\). The infinite series is then said to converge to \(f(x)\) in the interval \(|x| < a\). The difference
\[R_n(x) = f(x) - S_n(x),\]
the “remainder” of the series, measures the precision with which \(f(x)\) is approximated by the polynomial \(S_n(x)\) at \(x\). For example,
\[(1b) \quad \frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + R_n(x),\]
where the remainder \(R_n(x) = x^{n+1}/(1-x)\) tends to zero for \(|x| < 1\) as \(n\) increases; thus the infinite geometric series \(\sum_{v=0}^{\infty} x^v = 1/(1-x)\) results. To find simple manageable estimates for \(R_n\) in specific cases is a task of both theoretical and practical importance.

In this chapter we are concerned with such expansions for a wide class of functions, including all the “elementary” transcendental functions. It is a striking fact that in these expansions of transcendental functions the coefficients are elegant expressions in terms of integers. The approach to these expansions will be by Taylor’s theorem; later in Chapter 7 we shall discuss a different approach by a direct study of power series.

It should be emphasized that often just as for the geometrical series of Eq. (1a), the infinite expansion is not valid outside some interval