We now come to a situation where the natural way to define a function is not through a power series but through an integral depending on a parameter. We shall give a natural condition when we can differentiate under the integral sign, and we can then use Goursat's theorem to conclude that the holomorphic function so defined is analytic.

We shall be integrating over intervals. For concreteness let us assume that we integrate on $[0, \infty[$. A function $f$ on this interval is said to be **absolutely integrable** if

$$
\int_0^\infty |f(t)| \, dt
$$

exists. If the function is continuous, the integral is of course defined as the limit

$$
\lim_{B \to \infty} \int_0^B |f(t)| \, dt.
$$

We shall also deal with integrals depending on a parameter. This means $f$ is a function of two variables, $f(t, z)$, where $z$ lies in some domain $U$ in the complex numbers. The integral

$$
\int_0^\infty f(t, z) \, dt = \lim_{B \to \infty} \int_0^B f(t, z) \, dt
$$

is said to be **uniformly convergent** for $z \in U$ if, given $\varepsilon$, there exists $B_0$ such that if $B_0 < B_1 < B_2$, then

$$
\left| \int_{B_1}^{B_2} f(t, z) \, dt \right| < \varepsilon.
$$
The integral is absolutely and uniformly convergent for \( z \in U \) if this same condition holds with \( f(t, z) \) replaced by the absolute value \( |f(t, z)| \).

**XV, §1. THE DIFFERENTIATION LEMMA**

**Lemma 1.1.** Let \( I \) be an interval of real numbers, possibly infinite. Let \( U \) be an open set of complex numbers. Let \( f = f(t, z) \) be a continuous function on \( I \times U \). Assume:

(i) For each compact subset \( K \) of \( U \) the integral

\[
\int_I f(t, z) \, dt
\]

is uniformly convergent for \( s \in K \).

(ii) For each \( t \) the function \( z \mapsto f(t, z) \) is analytic. Let

\[
F(z) = \int_I f(t, z) \, dt.
\]

Then \( F \) is analytic on \( U \), \( D_2 f(t, z) \) satisfies the same hypotheses as \( f \), and

\[
F'(z) = \int_I D_2 f(t, z) \, dt.
\]

**Proof.** Let \( \{I_n\} \) be a sequence of finite closed intervals, increasing to \( I \). Let \( D \) be a disc in the \( z \)-plane whose closure is contained in \( U \). Let \( \gamma \) be the circle bounding \( D \). Then for each \( z \) in \( D \) we have

\[
f(t, z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(t, \zeta)}{\zeta - z} \, d\zeta,
\]

so

\[
F(z) = \frac{1}{2\pi i} \int_I \int_{\gamma} \frac{f(t, \zeta)}{\zeta - z} \, d\zeta \, dt.
\]

If \( \gamma \) has radius \( R \), center \( z_0 \), consider only \( z \) such that \( |z - z_0| \leq R/2 \). Then

\[
\left| \frac{1}{\zeta - z} \right| \leq 2/R.
\]

For each \( n \) we can define

\[
F_n(z) = \frac{1}{2\pi i} \int_{I_n} \int_{\gamma} \frac{f(t, \zeta)}{\zeta - z} \, d\zeta \, dt.
\]