At the turn of the century, Hadamard and de la Vallee Poussin independently gave a proof of the prime number theorem, exploiting the theory of entire functions which had been developed by Hadamard. Here we shall give D. J. Newman's proof, which is much shorter. I have also benefited from Korevaar's exposition. See:


Newman's proof illustrates again several techniques of complex analysis: contour integration, absolutely convergent products in a context different from Weierstrass products, and various aspects of entire functions in a classical context. Thus this chapter gives interesting more advanced reading material, and displays the versatility of applications of complex analysis.

In Chapter XV we touched already on the zeta function, and gave a method to prove the functional equation and the analytic continuation to the whole plane by means of the Hankel integrals. Here we shall develop whatever we need of analysis from scratch, but we assume the unique factorization of integers into primes.

We recall the notation: if \( f, g \) are two functions of a variable \( x \), defined for all \( x \) sufficiently large \( x \), and \( g \) is positive, we write

\[ f = O(g) \]

to mean that there exists a constant \( C > 0 \) such that \( |f(x)| \leq Cg(x) \) for
Let $s$ be a complex variable. For $\Re(s) > 1$ the series
\[ \sum_{n=1}^{\infty} \frac{1}{n^s} \]
converges absolutely, and uniformly for $\Re(s) \geq 1 + \delta$, with any $\delta > 0$. One sees this by estimating
\[ \left| \frac{1}{n^s} \right| \leq \frac{1}{n^{1+\delta}} \]
and by using the integral test on the real series $\sum 1/n^{1+\delta}$, which has positive terms.

As you should know, a prime number is an integer $\geq 2$ which is divisible only by itself and 1. Thus the prime numbers start with the sequence 2, 3, 5, 7, 11, 13, 17, 19, …. 

**Theorem 1.1.** The product
\[ \prod_p \left( 1 - \frac{1}{p^s} \right) \]
converges absolutely for $\Re(s) > 1$, and uniformly for $\Re(s) \geq 1 + \delta$ with $\delta > 0$, and we have
\[ \zeta(s) = \prod_p \left( 1 - \frac{1}{p^s} \right)^{-1}. \]

**Proof.** The convergence of the product is an immediate consequence of the definition given in Chapter X, §1 and the same estimate which gave the convergence of the series for the zeta function above. In the same region $\Re(s) \geq 1 + \delta$, we can use the geometric series estimate to conclude that
\[ \left( 1 - \frac{1}{p^s} \right)^{-1} = 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \cdots = E_p(s), \text{ say.} \]

Using a basic fact from elementary number theory that every positive integer has unique factorization into primes, up to the order of the