CHAPTER VII  
The Gamma function

§ 1. THE GAMMA FUNCTION IN THE REAL DOMAIN

1. DEFINITION OF THE GAMMA FUNCTION

We have defined (Set Theory, III, p. 179) the function \( n! \) for every integer \( n \geq 0 \), as equal to the product \( \prod_{0 \leq k < n} (n - k) \); so \( 0! = 1 \) and \( (n + 1)! = (n + 1) n! \) for \( n \geq 0 \).

We set \( \Gamma(n) = (n - 1)! \) for each integer \( n \geq 1 \); we propose to define, on the set of real numbers \( x > 0 \), a continuous function \( \Gamma(x) \) extending the function \( \Gamma \) defined on the set of integers \( \geq 1 \).

It is clear that there are infinitely many such functions; since \( \Gamma(n + 1) = n \Gamma(n) \) for every integer \( n \geq 1 \) we shall restrict ourselves to considering, among the continuous functions that extend \( \Gamma \), those which satisfy the equation

\[
f(x + 1) = x f(x)
\]  

(1)

for every \( x > 0 \).

For a solution of this equation to be an extension of \( \Gamma(n) \) it is necessary and sufficient that it also satisfies \( f(1) = 1 \).

If \( f \) satisfies (1) then, by recursion on \( n \),

\[
f(x + n) = x(x + 1)(x + 2) \ldots (x + n - 1) f(x)
\]  

(2)

for every integer \( n > 1 \) and for all \( x > 0 \). This relation shows, in particular, that the values of \( f \) on an interval \([n, n + 1]\) (\( n \) an integer \( \geq 1 \)) are determined by its values on the interval \([0, 1]\). Conversely, let \( \varphi \) be a continuous function on \([0, 1]\) satisfying only the conditions \( \varphi(1) = 1 \), \( \lim_{x \to 0} x \varphi(x) = 1 \); for every integer \( n \geq 1 \) let us define \( f \) on the interval \([n, n + 1]\) by the relation

\[
f(x) = (x - 1)(x - 2) \ldots (x - n) \varphi(x - n);
\]

it is clear that \( f \) is continuous on \([0, +\infty[\), satisfies the equation (1), and extends \( \Gamma(n) \).

If \( f \) is a continuous solution of (1) and takes values > 0 on \([0, 1]\) it takes values > 0 on \([0, +\infty[\), by (2); the function \( g(x) = \log f(x) \) is then defined and continuous.
on $[0, +\infty[$ and satisfies the equation
\[
g(x + 1) - g(x) = \log x
\]
on this interval.

If $g_1$ is a second continuous solution of (3) on $[0, +\infty[$, and if $h = g_1 - g$, then one has $h(x + 1) - h(x) = 0$ for every $x > 0$; in other words, $h$ is a continuous periodic function of period 1, defined on $[0, +\infty[$; conversely, for every $h$ of this nature, $g + h$ is a continuous solution of (3).

**Proposition 1.** There exists one and only one convex function $g$ defined on $[0, +\infty[$ that satisfies the equation (3) and takes the value 0 for $x = 1$.

First we show that if there is a function $g$ satisfying the conditions stated then it is well-determined on the interval $[0, 1]$, and consequently on the interval $[0, +\infty[$. Indeed, for every integer $n > 1$ the gradient of the line joining the point $(n, g(n))$ to the point $(x, g(x))$ is an increasing function of $x$, since $g$ is convex (I, p. 27, prop. 5); one thus must have, for $0 < x \leq 1$,
\[
\frac{g(n - 1) - g(n)}{(n - 1) - n} \leq \frac{g(n + x) - g(n)}{(n + x) - n} \leq \frac{g(n + 1) - g(n)}{(n + 1) - n}
\]
that is, by (3),
\[
x \log(n - 1) \leq g(x + n) - g(n) \leq x \log n.
\]
Now, by (3),
\[
g(x + n) - g(n) = g(x) + \log x + \sum_{k=1}^{n-1} (\log(x + k) - \log k).
\]
Moreover, one can write $\log n = \sum_{k=2}^{n} \log \frac{k}{k - 1}$ so the inequality (4) can be written
\[
x \sum_{k=2}^{n-1} \log \frac{k}{k - 1} \leq g(x) + \log x + \sum_{k=2}^{n} (\log(x + k - 1) - \log(k - 1))
\]
\[
\leq x \sum_{k=2}^{n} \log \frac{k}{k - 1}.
\]
Let us put, for every $n \geq 2$,
\[
u_n(x) = x \log \frac{n}{n - 1} - \log(x + n - 1) + \log(n - 1)
\]
and
\[
g_n(x) = - \log x + \sum_{k=2}^{n} u_k(x).
\]