ENO Schemes with Subcell Resolution*

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In this paper, we introduce the notion of subcell resolution, which is based on the observation that unlike point values, cell-averages of a discontinuous piecewise-smooth function contain information about the exact location of the discontinuity within the cell. Using this observation we design an essentially non-oscillatory (ENO) reconstruction technique which is exact for cell averages of discontinuous piecewise-polynomial functions of the appropriate degree. Later on we incorporate this new reconstruction technique into Godunov-type schemes in order to produce a modification of the ENO schemes which prevents the smearing of contact discontinuities. © 1989 Academic Press, Inc.

1. INTRODUCTION

In [6–8] we have introduced a class of essentially nonoscillatory (ENO) schemes that generalizes Godunov’s scheme [2] to a high order of accuracy.

In this paper we present a modification of the ENO schemes which is designed to prevent smearing of linear discontinuities. This is done by adding a correction term to the numerical flux of the ENO scheme. First, we shall derive a correction term to account for discontinuities in the scalar constant coefficient case. Later we shall apply the scalar correction to the linearly degenerate characteristic field in the Euler equations in order to improve the resolution of contact discontinuities.

Let \( \{I_j \times [t_n, t_{n+1})\} \), where \( I_j = [x_{j-1/2}, x_{j+1/2}] \), \( x = \frac{x}{h} \), \( t_k = kt \), be a partition of \( \mathbb{R} \times \mathbb{R}^+ \). Let \( \bar{u}_j^T \) be the “cell-average” of \( u \) at time \( t_n \), i.e.,

\[
\bar{u}_j^T = \frac{1}{h \int_{I_j}} u(x, t_n) \, dx. \tag{1.1}
\]

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The cell-average of the solution to the initial value problem
\[ u_t + f(u)_x = 0, \quad u(x, 0) = u_0(x) \] (1.2)
satisfies
\[ \tilde{u}^{n+1}_j = \tilde{u}^n_j - \lambda [ \tilde{f}(x_{j+1/2}, t_n; u) - \tilde{f}(x_{j-1/2}, t_n; u) ], \] (1.3a)
where \( \lambda = \tau/h \) and
\[ \tilde{f}(x, t; u) = \frac{1}{\tau} \int_0^\tau f(u(x, t + \eta)) \, d\eta. \] (1.3b)

The ENO schemes can be written in the standard conservation form
\[ v^{n+1}_j = v^n_j - \lambda (\tilde{f}_j-1/2 - \tilde{f}_j+1/2) \equiv [E_h(\tau) \cdot v^n]_j; \] (1.4a)
here \( E_h \) denotes the numerical solution operator and \( \tilde{f}_j+1/2 \), the numerical flux, denotes a function of \( 2k \) variables
\[ \tilde{f}_j+1/2 = f(v^n_{j-k+1}, ..., v^n_{j+k}) \] (1.4b)
which is consistent with the flux \( f(u) \) in (1.2), in the sense that \( \tilde{f}(u, u, ..., u) = f(u) \). Unlike standard difference schemes, \( v^n_j \) in the ENO schemes is a high-order approximation to the cell-average \( \tilde{u}^n_j \), and not to the point value \( u(x_j, t_n) \). Setting \( v^n_j = \tilde{u}^n_j \) in the numerical scheme (1.4) and comparing it to relation (1.3), we see that if the numerical flux \( \tilde{f}_j+1/2 = \tilde{f}(\tilde{u}^n_{j-k+1}, ..., \tilde{u}^n_{j+k}) \) can be expanded as
\[ \tilde{f}(\tilde{u}^n_{j-k+1}, ..., \tilde{u}^n_{j+k}) = \frac{1}{\tau} \int_0^\tau f(u(x_{j+1/2}, t_n + \eta)) \, d\eta + d(x_{j+1/2})h^r + O(h^{r+1}) \] (1.5a)
then the truncation error
\[ \tilde{u}^{n+1}_j - [E_h \cdot \tilde{u}^n]_j = \lambda [d(x_{j+1/2}) - d(x_{j-1/2})]h^r + O(h^{r+1}), \] (1.5b)
is \( O(h^{r+1}) \) wherever \( d(x) \) is Lipschitz continuous, i.e., the scheme (1.4) is \( r \)th order accurate in the sense of cell averages.

The most important ingredient in the ENO schemes is a procedure to reconstruct a piecewise-smooth function \( w(x) \) from its given cell-averages \( \{ \tilde{w}_j \} \). This reconstruction, which we denote by \( R(x; \tilde{w}) \), is a piecewise-polynomial function of \( x \) that has a uniform polynomial degree \( (r-1) \) and satisfies:

(i) at all points \( x \) for which there is a neighborhood where \( w \) is smooth
\[ R(x; \tilde{w}) = w(x) + e(x)h^r + O(h^{r+1}); \] (1.6a)