Exact Algorithms for Some Multi-level Location Problems on a Chain and a Tree

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Abstract

The multi-level Simple Plant Location Problem (MLP) are studied. Assumed that cost of transporting unit commodity from one site to other site equal to the sum of edge lengths in the path between these sites. Let \( p \) be a number of levels, \( n \) — a number of demand sites, \( m_r \) — a number of feasible facilities on level \( r, 1 \leq r \leq p \).

For the MLP on a chain exact algorithms \( \mathcal{A}_p \) and \( \tilde{\mathcal{A}}_p \) are constructed with time complexities \( (pnm_1 \cdots m_p) \) and \( (n^3 \sum_{i=1}^{p} m_r) \) respectively. For a case two and three levels algorithms \( \mathcal{A}_2 \) and \( \mathcal{A}_3 \) are preferable. In case \( p \geq 4 \) the polynomial algorithm \( \tilde{\mathcal{A}}_p \) is better.

For a case MLP on tree networks (in contrary to the chain problem) the optimal solution with connected regions of servicing can not exist (even in case two levels). Nevertheless in case two-level LP on tree networks we present the polynomial exact algorithm which requires \( (m^3 n) \) operations.

1 Introduction

Recently the researchers exhibit special interest in the Multi-level Simple Plant Location Problems (MLP) [1–6]. This class of problems can be characterized by presence of several levels of production in which raw material is processed before the finished product arrives to a consumer.

In general, this problem is \( NP \)-hard (even in the case of the one-level setting [7–8]). In [2–3] for solution of two- and multi-level location problems a branch and bound algorithm is suggested which in general is not effective [8]. Therefore, it is worth while to study special classes of the problem which can be solved by polynomial time algorithms [8–18].

In the present paper multi-level location problems on chain and two-level location problem on a tree-like network are considered provided that the transportation cost of the product unit from a site to a site is equal to the sum of lengths of edges in the subchain connecting these sites. The author is unaware of any effective algorithm for these cases. Note that the polynomial algorithm for solving MLP in a tree-like network which is suggested in [5] is incorrect even in the case of a chain.

In the second section the problem under study is formulated as a problem of linear integer programming (with using the Boolean variables for the choice and assigning). As far as research is concerned, another formulation of the problem is more appropriate in which the entries of the assignment matrix are considered as variables.

In section 3 we describe results for the multi-level simple plant location problem on chains (MLPC) [6]. The properties of optimal solutions of the MLPC are studied such as connectedness of service areas and consistency of location sites. These properties appeared to be very useful in constructing exact polynomial time algorithms for the standard (one-level) plant location problem [14–17] and the so-called standardization problem [8–13]. To find an exact solution of the one-level location problem on a tree-like network [14], on a 2-tree [17] and on \( k \)-tree [18], some algorithms with time complexity \( O(mn) \), \( O(m^3 n) \) and \( O(m^{k+1} n) \) are suggested, where \( n \) and \( m \) are the number of nodes (demand sites) and the number of possible plant location sites, respectively. For the MLP on a chain exact algorithms \( \mathcal{A}_p \) and \( \tilde{\mathcal{A}}_p \) are constructed with time complexities \( (pnm_1 \cdots m_p) \) and \( (n^3 \sum_{i=1}^{p} m_r) \).

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respectively. For a case two and three levels algorithms \( A_2 \) and \( A_3 \) are preferable. In case \( p \geq 4 \) the polynomial algorithm \( A_p \) is better.

In forth section we consider the two-level LP on tree-like networks. For a case MLP on tree networks (in contrary to the chain problem) the optimal solution with connected regions of servicing can not exist (even in case two levels). Nevertheless in case two-level LP on tree networks we present the polynomial exact algorithm which requires \( (m^3n) \) operations.

2 Statement of the Problem

The problem is formulated as follows. Let

\[ N = \{1, \ldots, n\} \] be a set of demand sites of the finished product,

\[ M_r \subset N \] be the set of possible plant location sites on level \( r \), \( 1 \leq r \leq p \);

\[ g_i^r \] be the cost of location of a plant on level \( r \) at site \( i, i \in M_r \), \( g_i^r \geq 0 \);

\[ c_{ij} \] be the transportation cost of the product unit from site \( i \) to site \( j \), where \( c_{ij} \geq 0 \) and \( i, j \in N \);

\[ b_j \] be the size of demand at site \( j \), where \( b_j > 0 \) and \( j \in N \).

It is assumed that each demand site of the finished product and each production site of any level receive the product only from one supplier; thus plant on level \( r \) receives the product from a plant on level \((r + 1), 1 \leq r \leq p - 1 \).

The problem of choosing subsets of location sites for each level (stage) \( I^r \subset M_r, r = 1, \ldots, p \), and assign the chosen plants to the demand sites so as to minimize the total costs of location of all chosen plants and of transportation of the product.

We first present a mathematical formulation of the plant location problem (LP) in the case of two levels with use made of the Boolean variables of choice \( x_i, y_k \) and assignment \( x_{kij} \), respectively:

\[
\sum_{i \in M_1} g_i^1 x_i + \sum_{k \in M_2} g_k^2 y_k + \sum_{j \in N} b_j \sum_{i \in M_1} \sum_{k \in M_2} (c_{hi} + c_{ij}) x_{kij}
\]

subject to

\[
\sum_{k \in M_2} \sum_{i \in M_1} x_{kij} = 1, j \in N,
\]

\[
\sum_{k \in M_2} x_{kij} \leq x_i, j \in N, i \in M_1,
\]

\[
\sum_{i \in M_1} x_{kij} \leq y_k, j \in N, k \in M_2,
\]

\[
x_i, y_k, x_{kij} \in \{0, 1\}.
\]

It will be more convenient to use another equivalent formulation using assignment vectors as variables as it was made for the one-level network location problem [14–16].

We introduce the following notation:

\[ \pi^r = (\pi^r_1, \ldots, \pi^r_n) \] is the plants assignment vector on level \( r \), where \( \pi^r_j \) is the number of sites in \( M_r \) in which the plant on level \( r \) serving demand site \( j \) is placed, \( r = 1, 2 \) and \( 1 \leq j \leq n \); \n
\[ \pi = (\pi^1, \pi^2) \] is a pair of assignment vectors;

\[ I^r(\pi) = \bigcup_{j \in N} \{ \pi^r_j \} \] is the set of plants on level \( r \) included in a solution \( \pi, r = 1, 2 \);

\[ Y^r(\pi) \] is the service area of plant \( i \) on level \( r \), i.e., the union over all \( j \) such that \( \pi^r_j = i \), where \( i \in M_r \) and \( r = 1, 2 \).

It is clear that \( I^r(\pi) \subset M_r \) and \( \bigcup Y^r(\pi) = N \), where the union is taken over all \( i \in I^r(\pi), r = 1, 2 \).

The two-level LP in the terms of the variables \( \pi^r_j \) can be written in a more compact form: find the minimum of the function

\[
\sum_{i \in I^r(\pi)} g_i^1 + \sum_{k \in I^r(\pi)} g_k^2 + \sum_{j \in N} b_j (c_{\pi^r_j j} + c_{\pi^r_j j}).
\]