Chapter 1
Basic Principles of Classical Mechanics

To describe the motion of mechanical systems one uses a variety of mathematical models which are based on different "principles" — laws of motion. In this chapter we list the basic objects and principles of classical mechanics. The simplest and most important model of motion of real bodies is Newtonian mechanics, which describes the motion of a free system of interacting point masses in three-dimensional Euclidean space. In §6 we discuss the extent to which Newtonian mechanics is useful in describing complicated models of motion.

§1. Newtonian Mechanics

1.1. Space, Time, Motion. Space, in which motion takes place, is three-dimensional and Euclidean, with a fixed orientation. We shall denote it by $E^3$. Fix a point $o \in E^3$ — an “origin” or “reference point”. Then the position of each point $s$ in $E^3$ is uniquely specified by its position (radius) vector $\vec{r}$ (with its tail and tip at $o$ and $s$, respectively). The set of all position vectors is the three-dimensional linear space $\mathbb{R}^3$. This space is equipped with the scalar product $\langle \cdot, \cdot \rangle$.

Time is one-dimensional; we denote it uniformly by $t$. The set $R = \{t\}$ is called the time axis.

A motion (or path) of the point $s$ is a smooth mapping $\Delta \rightarrow E^3$, where $\Delta$ is a time interval. We say that the motion is defined on the interval $\Delta$. To each motion there corresponds a unique smooth vector-function $r: \Delta \rightarrow \mathbb{R}^3$.

The velocity $v$ of the point $s$ at time $t \in \Delta$ is the derivative $dr/dt = \dot{r}(t) \in \mathbb{R}^3$. Velocity is clearly independent of the choice of the reference point.
The acceleration of the point $s$ is the vector $a = \dot{v} = \ddot{r} \in R^3$. It is customary to represent the velocity and acceleration as vectors with tail at the point $s$.

The set $E^3$ is also known as the position (or configuration) space of the point $s$. The pair $(s, v)$ is called a state of $s$, and the space $E^3 \times R^3 \{v\}$ is the state space (or the velocity phase space).

Now consider the more general case in which $n$ points $s_1, \ldots, s_n$ are moving in $E^3$. The set $E^{3n} = E^3 \{s_1\} \times \ldots \times E^3 \{s_n\}$ is called the position (configuration) space of this "free" system. In case it is necessary to exclude collisions of points, we must reduce $E^{3n}$ by removing the diagonal

$$A = \bigcup_{i < j} \{s_i = s_j\}.$$ 

Let $(r_1, \ldots, r_n) = r \in R^{3n}$ be the position vectors of the points $s_1, \ldots, s_n$. The motion of the free system is described by smooth vector-functions $r(t) = (r_1(t), \ldots, r_n(t))$. As above, we define the velocity

$$v = \dot{r} = (\dot{r}_1, \ldots, \dot{r}_n) = (v_1, \ldots, v_n) \in R^{3n}$$

and the acceleration

$$a = \ddot{r} = (\ddot{r}_1, \ldots, \ddot{r}_n) = (a_1, \ldots, a_n) \in R^{3n}.$$ 

The set $E^{3n} \times R^{3n} \{v\}$ is called the state space (or velocity phase space) of the system, and each pair $(s, v)$ is referred to as a state of the given system.

1.2. The Newton-Laplace Principle of Determinacy. This principle asserts that the state of a mechanical system at any fixed moment of time uniquely determines all of its (future and past) motion.

Suppose we know the state $(r_0, v_0)$ of the system at the moment of time $t_0$. Then by the principle of determinacy we also know the motion $r(t)$, with $r(t_0) = r_0$ and $\dot{r}(t_0) = \dot{r}_0 = v_0$, for all $t \in A \subset R$. In particular, we can calculate the acceleration $\ddot{r}$ at $t = t_0$. The result is $\ddot{r}(t_0) = f(t_0, r_0, \dot{r}_0)$, where $f$ is a function whose existence follows from the Newton-Laplace principle. Since we may choose an arbitrary value for $t_0$, we conclude that the equation

$$\ddot{r} = f(t, r, \dot{r})$$

holds for all $t$. This differential equation is known as the equation of motion or as Newton's equation. The existence of Newton's equation (with a smooth vector-function $f: R \{t\} \times R^{3n} \{r\} \times R^{3n} \{r\} \rightarrow R^{3n}$) and the determinacy principle are actually equivalent. This is a consequence of the theorem of existence and uniqueness of solutions from the theory of ordinary differential equations. The function $f$ in Newton's equation is usually determined experimentally. Its specification is part of the definition of the mechanical system under consideration.

We will now give examples of Newton's equation.

---

1 We will assume that all functions we encounter in dynamics are smooth.