Chapter 3
Symmetry Groups and Reduction
(Lowering the Order)

§1. Symmetries and Linear First Integrals

1.1. E. Noether's Theorem. Let \((M, L)\) be a Lagrangian system and \(v\) a smooth vector field on \(M\). The field \(v\) generates a one-parameter group of diffeomorphisms \(g^a: M \rightarrow M\), where \(g^a(x)\) is the solution of the differential equation

\[
\frac{d}{d\alpha} g^a(x) = v(g^a(x))
\]

with initial condition \(g^0(x) = x\).

**Definition.** The Lagrangian system \((M, L)\) admits the group \(g^a\) if the Lagrangian \(L\) is invariant under the maps \(g^*_a: TM \rightarrow TM\). The group \(g^a\) and the vector field \(v\) are naturally called a symmetry group and a symmetry vector field, respectively.

Let \(\gamma: \Delta \rightarrow M\) be a motion of the Lagrangian system \((M, L)\). Then \(g^a \circ \gamma: \Delta \rightarrow M\) is also a motion, for every value of \(\alpha\).

In the nonautonomous case \(L\) is a smooth function on the tangent bundle of the extended configuration space \('M = M \times \mathbb{R}\). We call the one-parameter group of diffeomorphisms \('g^a: 'M \rightarrow 'M\) a symmetry group of the system \((M, L)\) if \('g^a(x, t) = (y, t)\) for all \((x, t) \in M \times \mathbb{R}\) (i.e., \('g^a\) preserves time) and the maps \('g^*_a\) preserve \(L\). To the group \('g^a\) there corresponds the smooth vector field

\[
'v(x, t) = \frac{d}{d\alpha} ('g^a(x, t))|_{\alpha = 0}
\]

on \('M\). Obviously, \('v(x, t) = (v(x, t), 0) \in T_{(x, t)}(M \times \mathbb{R})\), and \(v(x, t)\) may be regarded as a vector field on \(M\) which depends smoothly on \(t\).

**Lemma 1.** System \((M, L)\) admits the symmetry group \(g^a\) if and only if

\[
(p \cdot v)' = [L] \cdot v.
\]
This is a consequence of the identity
\[ \frac{d}{dx} L(g_t x) = (L_x' \cdot v) - [L] \cdot v. \]

Lemma 1 is valid also in the nonautonomous case. From equality (2) we obtain

**Theorem 1.** If system \((M, L)\) admits the group \(g^a\), then \(\mathcal{J} = p \cdot v\) is a first integral of the equations of motion.18

Let \((M, \langle , \rangle, V)\) be a natural mechanical system. The Lagrangian \(L = \frac{1}{2} \langle \dot{x}, \dot{x} \rangle + V(x)\) is invariant under the action of the group \(g^a\) if and only if this property is shared by the Riemannian metric \(\langle , \rangle\) and the force function \(V\). For such a system the first integral \(\mathcal{J}\) is obviously equal to \(\langle v, x \rangle\), and hence it is linear in the velocity.

**Example 1.** Suppose that in some (local) system of coordinates \(x_1, \ldots, x_n\) on \(M\) the Lagrangian \(L\) does not depend on \(x_1\). Then the system \((M, L)\) admits locally the symmetry group \(g^a: x_1 \to x_1 + \alpha, x_k \to x_k (k \geq 2)\). The vector field corresponding to this group is \(v = \frac{\partial}{\partial x_1}\). Theorem 1 asserts that the quantity \(\mathcal{J} = p \cdot v = p_1 = L_{x_1}\) is a first integral. In mechanics \(x_1\) is called a cyclic coordinate, and \(\mathcal{J}\) a cyclic integral. In particular, the energy integral is a cyclic integral of a suitable extended Lagrangian system. To show this, we introduce a new time variable \(\tau\) by the rule \(t = t(\tau)\), and define the function \(\mathcal{L}: T'M \to \mathbb{R}\) \(\forall M = M \times \mathbb{R}\) by the formula
\[ \mathcal{L}(x', t', x, t) = L(x'/t', x, t)t', \quad (\cdot)' = \frac{d}{d\tau} (\cdot). \]

It follows from Hamilton's variational principle and the equality
\[ \int_{\tau_1}^{\tau_2} \mathcal{L} d\tau = \int_{\tau_1}^{\tau_2} L dt \]
that if \(x: [\tau_1, \tau_2] \to M\) is a motion of the system \((M, L)\), then \((x, t): [\tau_1, \tau_2] \to T'M\) is a motion of the extended Lagrangian system \((M', \mathcal{L})\). In the autonomous case time \(t\) is a cyclic coordinate and the cyclic integral
\[ \frac{\partial \mathcal{L}}{\partial t'} = L - \frac{\partial L}{\partial \dot{x}} \dot{x} = \text{const.} \]

coincides with the energy integral. \(\triangle\)

**Theorem 2.** If \(v(x_0) \neq 0\), then in a small neighborhood of the point \(x_0\) there exist local coordinates \(x_1, \ldots, x_n\) such that \(\mathcal{J} = p \cdot v = p_1\).

18 Theorem 1 was first formulated in this form by E. Noether in 1918. Lagrange and Jacobi were already aware of the connection between the conservation of momentum and angular momentum, and the groups of translations and rotations, respectively. Theorem 1 for natural systems was published by M. Lévy in 1878.